Lecture 7: Sampling Distributions

Monte Carlo Simulation · CLT in Action · Standard Errors

Previously, on Lectures 4–6. . .

MLE: $\hat{\theta} = \arg\max \ell(\theta)$. Find the parameter that makes data most likely.

MAP: $\hat{\theta} = \arg\max[\ell(\theta) + \log P(\theta)]$. MLE with a prior = regularization.

Fisher info: $I(\theta) = -E[\ell''(\theta)]$. Curvature of the log-likelihood.

CR bound: $\mathrm{Var}(\hat{\theta}) \geq 1/(nI(\theta))$. A floor on how precise we can be.

Asymptotic normality: $\hat{\theta}_{\mathrm{MLE}} \dot{\sim} N(\theta_0,\ 1/(nI(\theta_0)))$ for large n.

5 3.2

$X_1, X_2, \ldots$

$\tilde{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\boxed{3.2}$

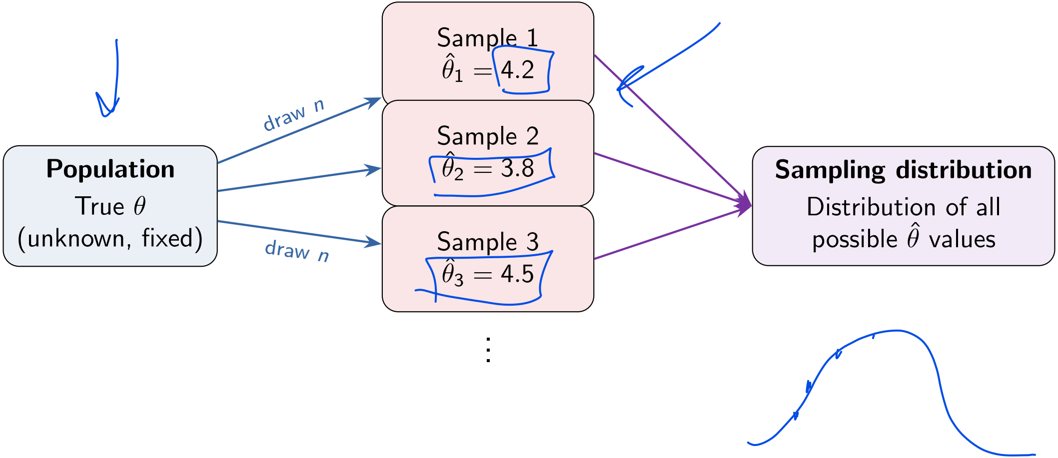
Today: You computed $\hat{\theta} = 3.2$. But how **reliable** is this number?

If you repeated the experiment, would you get 3.2 again? 3.1? 5.7?

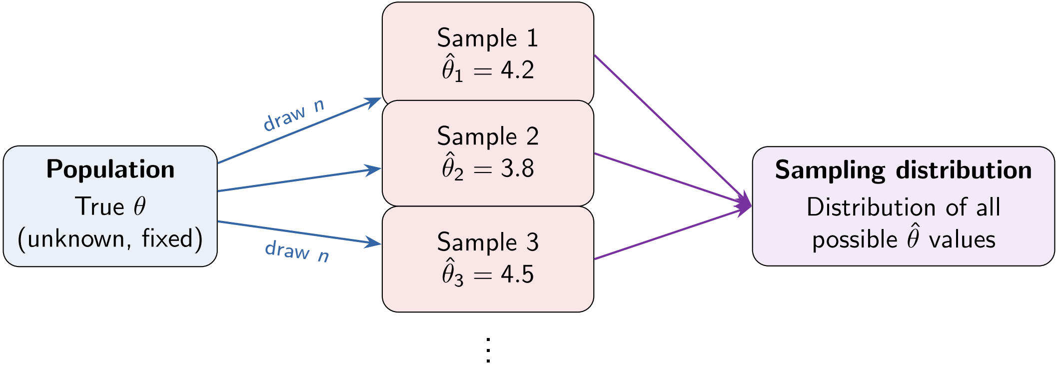
Your Estimator Is Random

Different sample $\Rightarrow$ different $\hat{\theta}$

The Thought Experiment



The Thought Experiment

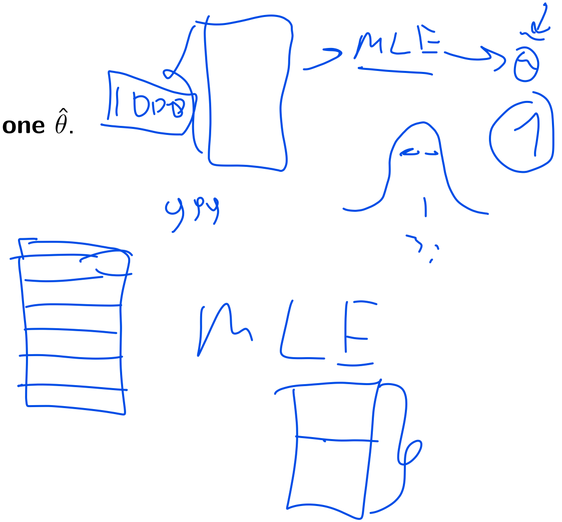


Key insight: $\hat{\theta}$ is a *function of random data*, so $\hat{\theta}$ itself is a random variable.

Its distribution is called the **sampling distribution**. Understanding it is the foundation of all inference.

But We Only Have One Sample!

In real life, you run the experiment **once**. You get **one** $\hat{\theta}$.



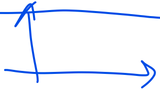
The problem:

We can't actually repeat the experiment 1,000 times to see the sampling distribution.

Today: solutions 1 and 2. Next lecture: solution 3 (the bootstrap).

But We Only Have One Sample!

$\boxed{y \sim 4X + 3}$ $\quad X \sim N$



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Three solutions:

1. **Theory** — derive it mathematically
2. **Simulation** — fake it on a computer
3. **Bootstrap** — resample the data itself

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Seeing It: Monte Carlo Simulation

Let the computer repeat the experiment for you

The Monte Carlo Recipe

$B(\theta)$

Setup: Pick a true population distribution F with known θ.

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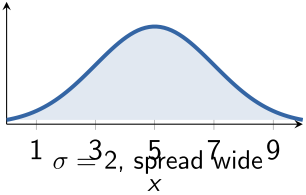
Example: Sample Mean from $N(5,4)$

Population: $X_i \sim N(\mu = 5, \sigma^2 = 4)$, $n = 10$, estimator: $\hat{\mu} = \bar{X}$

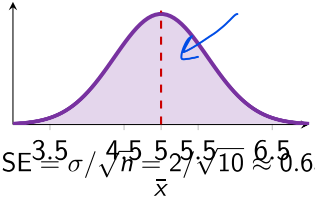
Population $N(5,4)$



$\sigma = 2$, spread wide

simulate $R = 10{,}000$

Sampling dist. of $\bar{X}$

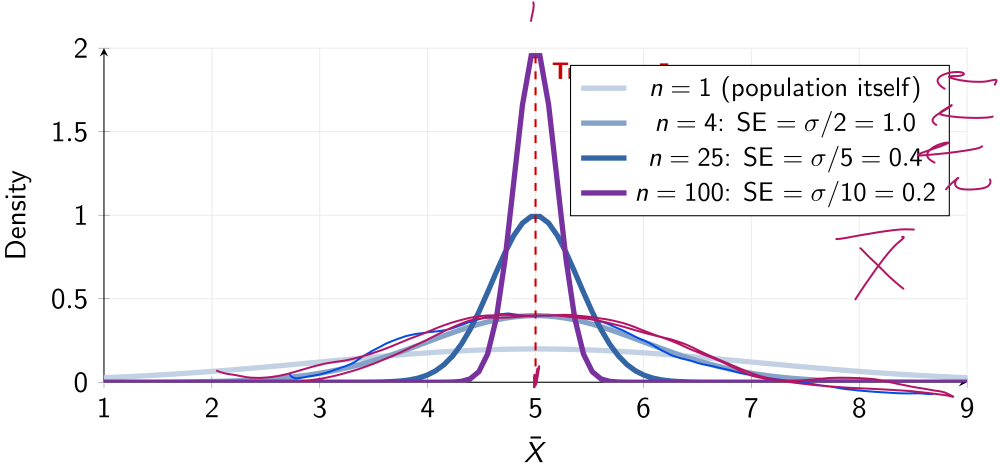


$\text{SE} = \sigma/\sqrt{n} = 2/\sqrt{10} \approx 0.63$

$\sigma / \sqrt{n}$

The sampling distribution is **much narrower** than the population — averaging reduces the spread by $\sqrt{n}$.

What Happens as n Grows?

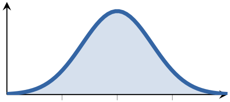


More data $\Rightarrow$ the sampling distribution **concentrates** around the true value. This is **consistency**.

It Works for Any Estimator

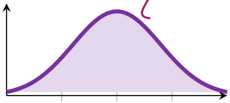
Not just $\bar{X}$! **Every** estimator has a sampling distribution.

Sample Mean $\bar{X}$



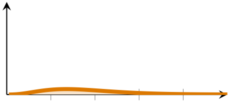
Symmetric, bell-shaped

Sample Median



Also bell-shaped, but **wider**

Sample Variance S^2



Right-skewed for small n

(follows a scaled χ^2_{n-1})

Comparing Estimators via Their Sampling Distributions

Key observation

Observation: Different estimators have **different** sampling distributions.

The mean is narrower than the median (for Normal data) — it's a *more efficient* estimator.

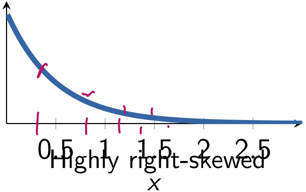
Recall from Lecture 4: the Cramér–Rao bound tells us the *best possible* variance.

Coming up: We'll see that the **Central Limit Theorem** explains *why* sampling distributions tend to look Normal — and **Fisher information** determines how narrow they are.

Example: MLE for Exp(λ)

Population: $X_i \sim \text{Exp}(\lambda = 2)$, MLE: $\hat{\lambda} = 1/\bar{X}$

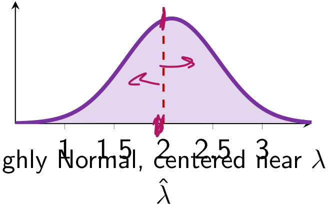
Population Exp(2)



Highly right-skewed

$R = 10{,}000$, $n = 20$

Sampling dist. of $\hat{\lambda}_{\text{MLE}}$



Roughly Normal, centered near $\lambda = 2$

The population is **exponential** (skewed), but the MLE is **approximately Normal**. Why? **CLT!**

The Central Limit Theorem

Why everything looks Normal

CLT: The Most Important Theorem in Statistics

Central Limit Theorem: If $X_1, \ldots, X_n$ are i.i.d. with mean μ and variance $\sigma^2 < \infty$, then

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty$$

Equivalently: $\bar{X} \dot{\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$ for large n.

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

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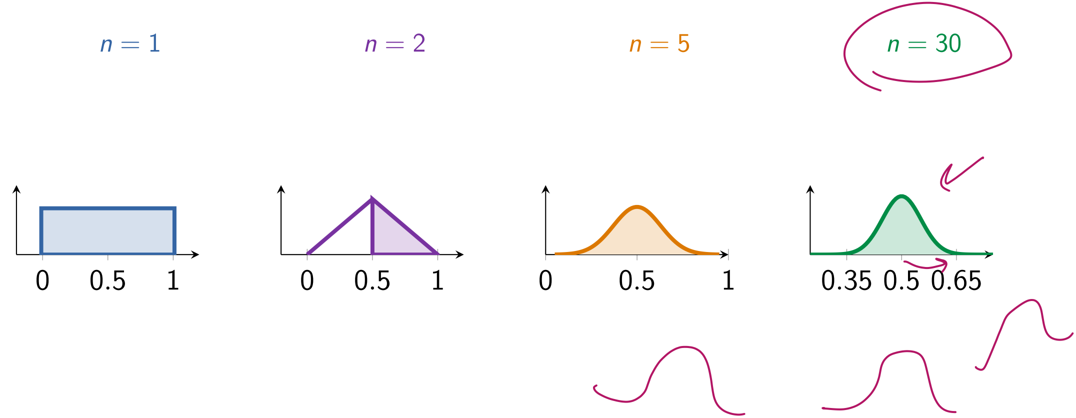
The magic: It doesn't matter what the population looks like — uniform, exponential, Bernoulli, bimodal, anything. As long as the variance is finite, $\bar{X}$ becomes Normal.

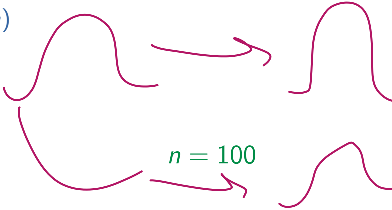
CLT in Action: Uniform Population

$X_i \sim \text{Uniform}(0, 1)$ — completely flat, yet $\bar{X}$ becomes Normal

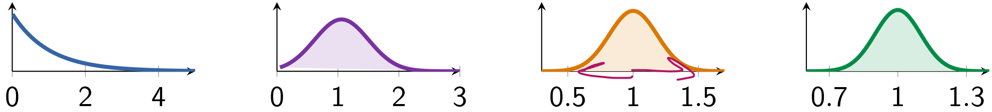


CLT in Action: Exponential Population

$X_i \sim \mathrm{Exp}(1)$ — strongly right-skewed, yet $\bar{X}$ becomes Normal (needs larger n)



$n = 1$ $n = 5$ $n = 30$ $n = 100$



How Large Must n Be?

Symmetric
(Uniform, Normal)
$n \geq 5$ is often enough

Mildly skewed
(Exponential, Poisson)
$n \geq$ 25–30 usually works

Heavily skewed
(Pareto, lognormal)
May need $n \geq 100$ or more

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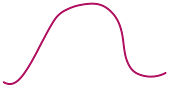

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The common "$n \geq 30$" rule is a useful heuristic, not a law.

For symmetric data, even $n = 5$ is fine. For heavy tails, even $n = 100$ may not suffice.

When in doubt — **simulate!** That's what Monte Carlo is for.

CLT: What It Does **Not** Say

The CLT is powerful, but often over-applied. Three common misconceptions:

✗ "With large n, the **data** becomes Normal."
✓ The data stays whatever it is. Only the *sample mean* (and sums) become Normal.

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✓ It requires **finite variance**. The **Cauchy** distribution has no mean or variance — averaging Cauchy data does *not* converge to Normal. Try it in the practical!

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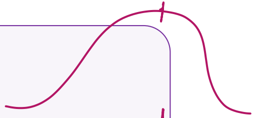
✗ "CLT works for any sample, even if observations are dependent."
✓ The basic CLT requires **independence** (or at least weak dependence). Time series, spatial data, and clustered samples need special CLT versions.

CLT for MLEs: Even More Powerful

The CLT doesn't just apply to $\bar{X}$. For **any** MLE (under regularity conditions):

$$\hat{\theta}_{\text{MLE}} \quad \dot{\sim} \quad N\left(\theta_0, \ \frac{1}{n \cdot I(\theta_0)}\right)$$

The MLE is approximately Normal, centered at the truth,
with variance determined by the **Fisher information**.



$$\frac{1}{nI(\theta_0)}$$

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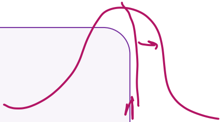

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$\frac{\lambda^2}{n}$

$N($

Bernoulli(p)	**Poisson(λ)**	**Exp(λ)**
$\hat{p} \dot{\sim} N\left(p, \ \frac{p(1-p)}{n}\right)$	$\hat{\lambda} \dot{\sim} N\left(\lambda, \ \frac{\lambda}{n}\right)$	$\hat{\lambda} \dot{\sim} N\left(\lambda, \ \frac{\lambda^2}{n}\right)$

Each follows from $I(\theta)$: Bernoulli $I(p) = 1/(p(1-p))$, Poisson $I(\lambda) = 1/\lambda$, etc.

Standard Error

The most important number after the estimate itself

SD $\neq$ SE: The Most Confused Pair in Statistics

Standard Deviation (SD)

Spread of the **data**

$$\sigma = \sqrt{E[(X - \mu)^2]}$$

"How variable are individual observations?"

Does not shrink with n.
More data doesn't make individuals less variable.

Standard Error (SE)

Spread of the **estimator**

$$\mathrm{SE}(\bar{X}) = \sigma/\sqrt{n}$$

"How variable is $\hat{\theta}$ across repeated experiments?"

Shrinks with n.
More data $\Rightarrow$ more precise estimate.

$$\frac{\sigma}{\sqrt{n}}$$

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Example: Heights of adults: SD $\approx 7\,\text{cm}$ (people vary).
Average of 100 heights: SE $= 7/\sqrt{100} = 0.7\,\text{cm}$ (the average is precise).

Standard Error of the Sample Mean

$$\mathrm{SE}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

The **standard error** is the standard deviation of the sampling distribution.

It measures how much $\bar{X}$ would vary if we repeated the experiment.

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Derivation:

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

$\bar{X}$

$\frac{\sigma}{n}$

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$$\Rightarrow \quad \mathrm{SE}(\bar{X}) = \sqrt{\mathrm{Var}(\bar{X})} = \frac{\sigma}{\sqrt{n}}$$

Uses i.i.d. assumption: observations are independent, so variances **add**.

SE in Practice: A Polling Example

Scenario: A poll of $n = 1{,}000$ voters finds $\hat{p} = 0.52$ support for a candidate.

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Step 1: Compute the estimated SE.

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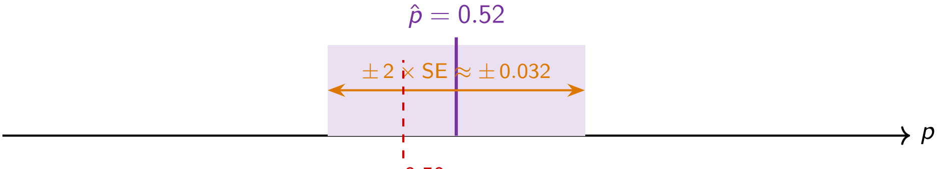
71-53

Step 2: Interpret. The estimate is $\hat{p} = 0.52 \pm 0.016$.



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0.008

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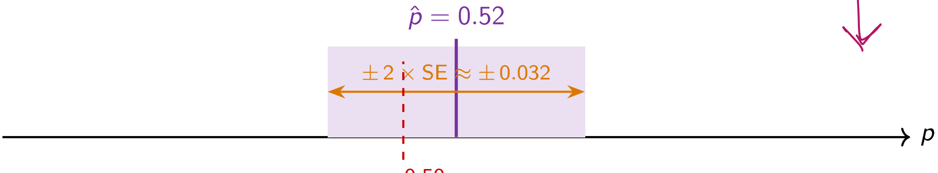
$\sqrt{n}$



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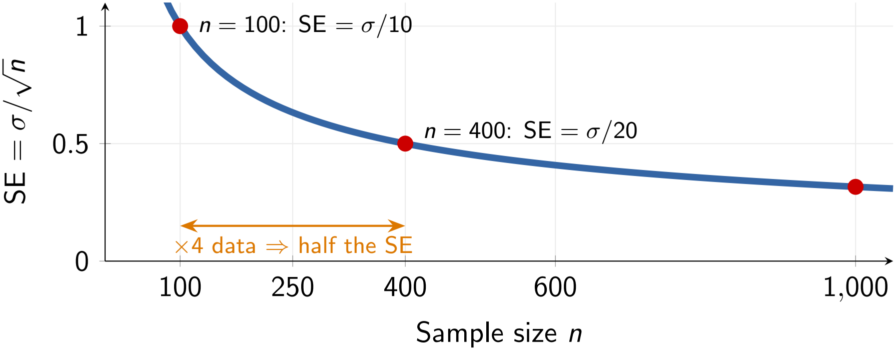
SE

Rule of thumb: always report $\hat{\theta} \pm \text{SE}$. An estimate without a standard error is like a measurement without units — it tells you nothing about reliability.

The $\sqrt{n}$ Law: Precision Is Expensive



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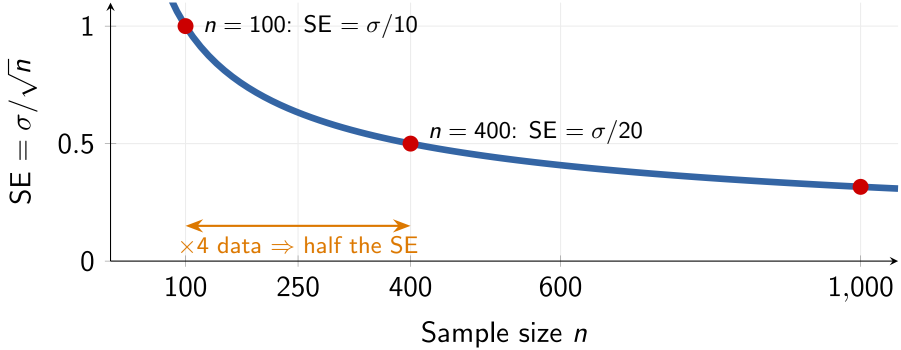

To **halve** the SE, you must **quadruple** the sample size.
Going from $n = 100$ to $n = 10{,}000$ only reduces SE by $10\times$.
This is why sample size planning matters!

SE for Any MLE: The General Formula

From Lecture 4 (Cramér–Rao) and Lecture 5 (MLE asymptotics):

$$\text{SE}(\hat{\theta}_{\text{MLE}}) \approx \frac{1}{\sqrt{n \cdot I(\theta)}}$$

$$\frac{1}{\sqrt{n I(\theta)}}$$

The Fisher information $I(\theta)$ determines the precision of the MLE.

More informative data (larger I) $\Rightarrow$ smaller SE.

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Estimated standard error — this is what we report in practice.

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Known vs Estimated SE

If σ is known:

$$\text{SE}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

Rare in practice.
Use the z-distribution (Normal).

$$\frac{\sigma}{\sqrt{n}}$$

But what *is* the t-distribution? And why do we need it?

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If σ is unknown (typical):

$$\widehat{\mathrm{SE}}(\bar{X}) = \frac{S}{\sqrt{n}}$$

Replace σ with $S = \sqrt{\frac{1}{n-1}\sum(X_i - \bar{X})^2}$.
Use the t-distribution.

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σ

$\left(\frac{1}{n-1}(X_i - \bar{X})\right)^2$

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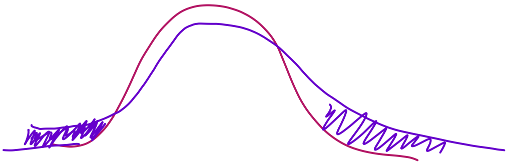
The t-Distribution: What It Is and Why It Exists

The problem: When σ is known, $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ exactly (for Normal data).

But when we **replace** σ **with** S, the ratio is *no longer* Normal — it has **heavier tails**:

$$T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$



The t-Distribution: What It Is and Why It Exists

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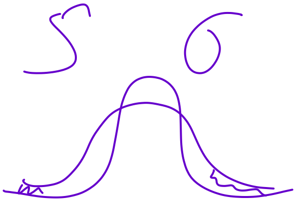
Why heavier tails? Sometimes S underestimates σ, making T too large. Sometimes it overestimates, making T too small. This **extra randomness from S** inflates the tails.



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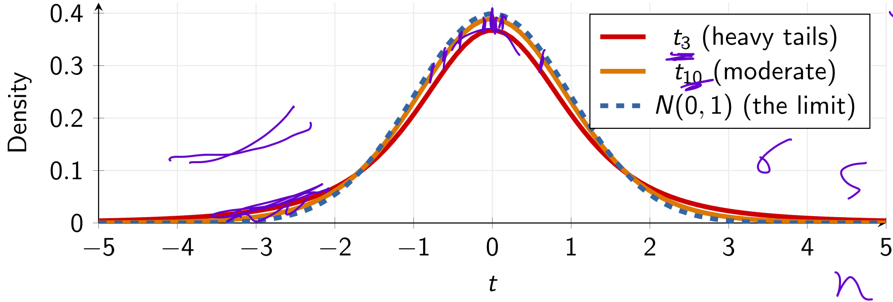
$\sqrt{\frac{1}{n-1}}$



n

Fewer observations (n small) $\Rightarrow$ S is noisy $\Rightarrow$ t_{n-1} has fat tails.

More observations (n large) $\Rightarrow$ $S \approx \sigma \Rightarrow t_{n-1} \to N(0,1)$. By $n \geq 30$, nearly identical.

The t-Distribution: Key Facts

Shape: Symmetric, bell-shaped, like a Normal but with **heavier tails**.

Parameter: Degrees of freedom $\nu = n - 1$. Smaller $\nu \Rightarrow$ heavier tails. $n-1$

Mean: 0 (for $\nu > 1$). **Variance:** $\nu/(\nu - 2)$ (for $\nu > 2$), slightly > 1. $n-2$

Convergence: As $\nu \to \infty$, $t_\nu \to N(0, 1)$. For $\nu \geq 30$, the difference is tiny. $n>2$

When to use: Normal data, unknown σ. This is the standard setting in practice.

3

1

2

3

Why "degrees of freedom" $= n - 1$?

We estimate σ using $S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$. The deviations from $\bar{X}$ must sum to 0, so only $n - 1$ of them are "free." This lost degree of freedom is exactly what makes T not Normal.

$X_1 + X_2 = 0$

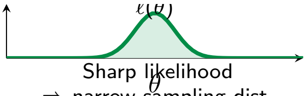
3 $\quad$ -3

Connecting the Dots

From data to uncertainty

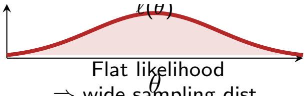
Fisher Information ↔ Sampling Distribution

High Fisher info

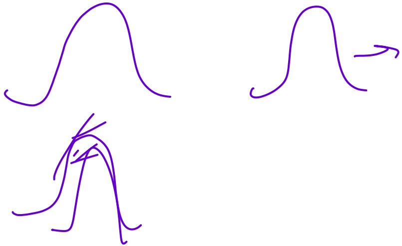


Sharp likelihood
$\Rightarrow$ narrow sampling dist.
$\Rightarrow$ small SE

Low Fisher info



Flat likelihood
$\Rightarrow$ wide sampling dist.
$\Rightarrow$ large SE



Fisher Information $\leftrightarrow$ Sampling Distribution

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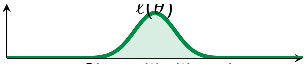

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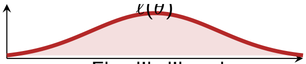

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The chain: More curvature ($I(\theta)$ $\uparrow$) $\Rightarrow$ Smaller variance ($1/(nI)$ $\downarrow$) $\Rightarrow$ Smaller SE $\Rightarrow$ More precise $\hat{\theta}$

This is why Fisher information is called "information" — it literally measures how much the data tells us about θ.

The Full Picture

$$\begin{tikzcd}[column sep=large]
\begin{array}{c}\textbf{Population}\\ \text{True } \theta\end{array} \arrow[r, "\text{i.i.d. draw}"] & \begin{array}{c}\textbf{Sample}\\ X_1, \ldots, X_n\end{array} \arrow[r, "\text{compute}"] & \begin{array}{c}\textbf{Estimator}\\ \hat{\theta} = T(\mathbf{X})\end{array} \arrow[r, "\text{repeat}"] & \begin{array}{c}\textbf{Sampling dist.}\\ \text{of } \hat{\theta}\end{array}
\end{tikzcd}$$

Two paths to quantify uncertainty: **analytical** (formulas, today) and **computational** (bootstrap, next lecture).

Preview: Since $\hat{\theta} \dot{\sim} N(\theta, \mathrm{SE}^2)$, about **68%** of estimates fall within $\pm 1\,\mathrm{SE}$, and **95%** within $\pm 1.96\,\mathrm{SE}$ of the truth. That's where confidence intervals come from — next lecture!

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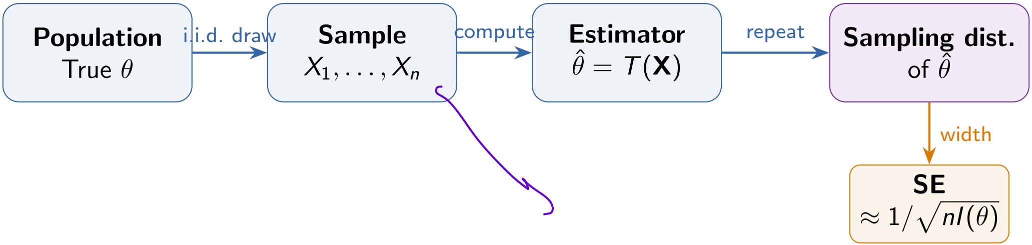

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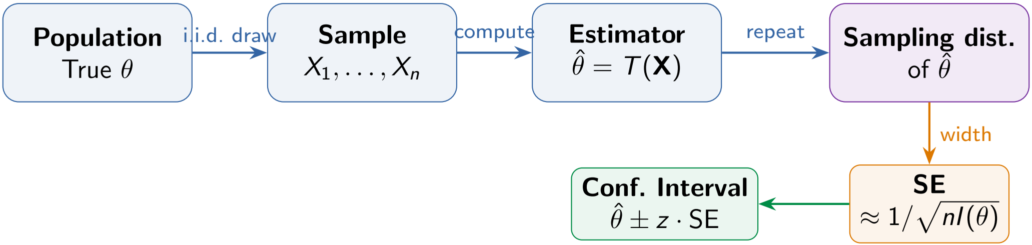

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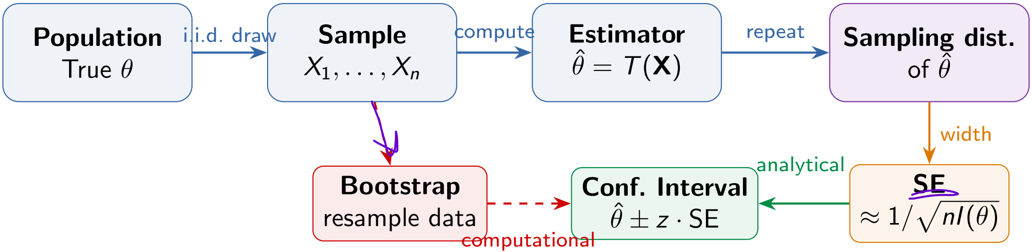

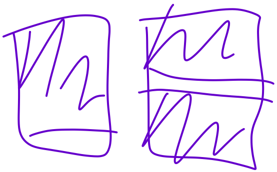

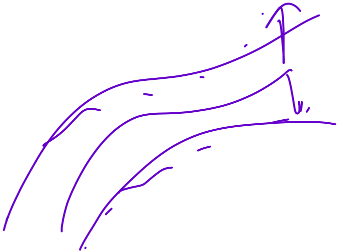

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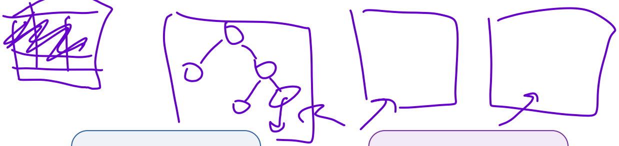

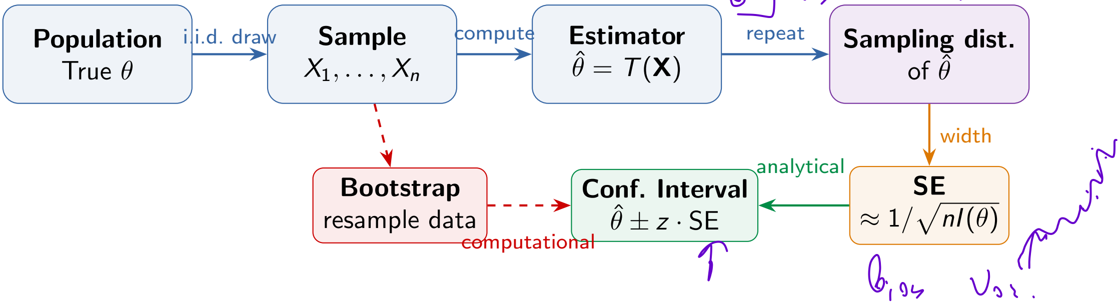

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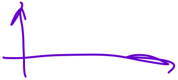

Recommended Visualizations & Resources

Interactive: Seeing Theory — Basic Probability (Brown University)

`seeing-theory.brown.edu/basic-probability` — drag sliders to change n and watch the sampling distribution form in real time.

Interactive: CLT Visualization (Online Stat Book)

`onlinestatbook.com/stat_sim/sampling_dist` — choose any population shape and see the sampling distribution for different n.

Video: 3Blue1Brown — But what is the Central Limit Theorem?

Beautiful visual explanation of *why* CLT works, using convolutions. Watch after the lecture.

Video: StatQuest — Standard Error

Clear explanation of the difference between SD and SE, with worked examples.

Reading: Wasserman, "All of Statistics" — Ch. 3 & 5

Convergence, CLT, and the delta method. Concise, mathematically clean, excellent reference.

Questions?

Next: Lecture 8 — Confidence intervals and the bootstrap