Lecture 1: Foundations

Probability vs Statistics · Population & Sample · i.i.d. · Plug-in Principle · Loss & Risk

How much should you trust a number?

A poll says: "52% support candidate A" $(n = 1{,}000)$

A clinical trial says: "Drug B reduces symptoms by 15%" $(n = 200)$

How confident should we be?

This entire course is about answering this question rigorously.

Probability vs Statistics

Rules (Model) $\xrightarrow{\textbf{Probability}}$ **Predictions (Data)** *forward problem*

fair die, $p = 1/6$

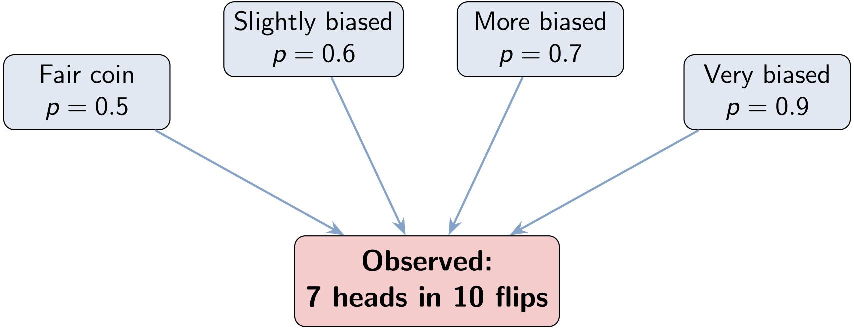
how often each face?

Observations (Data) $\xrightarrow{\textbf{Statistics}}$ **Rules (Model)** *inverse problem*

rolled 100 times, saw data

is the die fair?

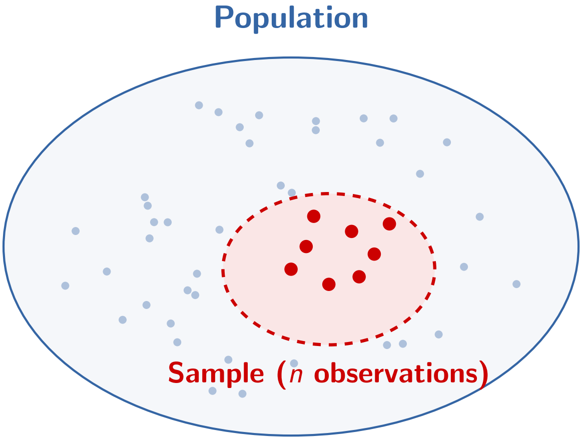
Why the inverse problem is harder



Many different models could have produced this data!

The inverse problem is **ill-posed** — statistics gives us tools to navigate this.

Population vs Sample



Population:
All units of interest

Can be finite or conceptually infinite

Sample:
The subset we actually observe

Parameter vs Statistic

Parameter θ

Fixed, unknown number
Describes the **population**

Examples:
μ = true mean lifetime
p = true approval rate
σ^2 = true variance

we estimate this

using this

Statistic $T(X_1, \ldots, X_n)$

Random variable, computable
Computed from the **sample**

Examples:
$\bar{X}$ = sample mean
$\hat{p}$ = sample proportion
S^2 = sample variance

A **parameter** is a fixed number. A **statistic** is a random variable.
Confusing these is the source of most beginner mistakes.

The Triple: Estimand / Estimator / Estimate

Estimand

The *target*

θ (true value)

"true mean height"

approximated by

Estimator

The *recipe*

$\bar{X} = \frac{1}{n}\sum X_i$

"sample mean formula"

applied to data

Estimate

The *number*

$\bar{x} = 172.4$ cm

"the value we got"

Discussion

A polling agency surveys 1,000 people and reports:

"62% support policy X"

Identify each:

1. What is the **population**?
2. What is the **parameter**?
3. What is the **sample**?
4. What is the **statistic**?
5. What is the **estimate**?

Discussion: Answers

"62% support policy X" $(n = 1{,}000)$

1. **Population:** all citizens of the country (eligible voters)
2. **Parameter:** $p =$ true proportion who support policy X (unknown)
3. **Sample:** the 1,000 people surveyed
4. **Statistic (estimator):** $\hat{p} = \frac{\#\ \text{who said ``yes''}}{n}$ (the formula/recipe)
5. **Estimate:** $\hat{p} = 0.62$ (the specific number from this sample)

The i.i.d. Assumption

Classical statistics assumes our sample $X_1, X_2, \ldots, X_n$ is **i.i.d.**:

Independent

Knowing X_1 tells you nothing about X_2

Each observation is a fresh draw

Identically Distributed

Every X_i comes from the same distribution F

Same process generates each one

When does i.i.d. hold?

- ✓ Random sampling from a large population
- ✓ Repeated independent measurements of the same quantity
- ✓ Controlled experiments with proper randomization

i.i.d. is an **idealization** — it's approximately true in many practical settings, and most of what we'll do this course assumes it.

When does i.i.d. break?

Time dependence
stock prices, weather

Non-response bias
who refuses the survey?

Spatial correlation
neighboring sensors

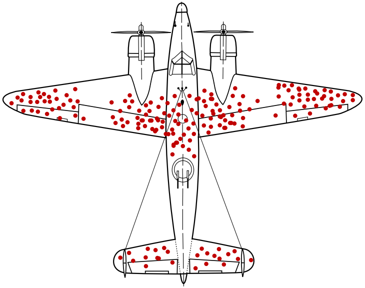
Distribution shift
training data $\neq$ deployment

Selection bias
hospital-only patients

Clustering
students within schools

Not a disaster — just means you need different tools.
But if you *pretend* non-i.i.d. data is i.i.d.,
your conclusions can be **wildly wrong**.

Survivorship Bias



WW2: Engineers studied bullet holes on returning bombers and proposed armoring the hit areas.

Abraham Wald: *"You're only seeing planes that* **survived***. Armor the places with* **no** *holes — those hits brought planes down."*

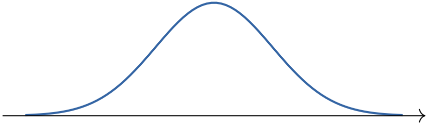
More examples:

- Online survey: "Do you have internet?" — 100% say yes
- "Soviet products lasted forever" — you only see the ones that survived
- Bus fare survey: asking people *on the bus* "100→150 AMD?" — only sampling current riders

The Plug-in Principle

Idea: We don't know the true distribution F, so replace it with the **empirical distribution** $\hat{F}_n$.

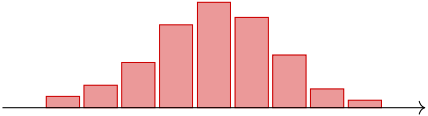
True distribution F
(unknown)



smooth, continuous

replace with $\longrightarrow$

Empirical distribution $\hat{F}_n$
(computable from data)



mass $1/n$ on each point

Plug-in in Action

Replace the **population quantity** with its **sample analogue**:

Want	**Population**	**Plug-in**
Mean	$\mu = \mathbb{E}_F[X]$	$\hat{\mu} = \bar{X} = \frac{1}{n}\sum X_i$
Variance	$\sigma^2 = \mathrm{Var}_F(X)$	$\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$
CDF	$F(t) = P(X \leq t)$	$\hat{F}_n(t) = \frac{\#\{X_i \leq t\}}{n}$

Glivenko–Cantelli theorem: $\hat{F}_n \to F$ uniformly as $n \to \infty$.
(The "fundamental theorem of statistics" — connects to LLN from Module 20.)

The Summarization Problem

You must summarize a distribution with a **single number** a.

How do you choose?

It depends on what "error" means to you.

This is formalized by a **loss function** $L(\theta, a)$.

Three Losses, Three Optimal Summaries

Squared Error

$L = (\theta - a)^2$

Penalizes large errors heavily

$\Downarrow$

Mean

Absolute Error

$L = |\theta - a|$

Linear penalty, robust to outliers

$\Downarrow$

Median

0–1 Loss

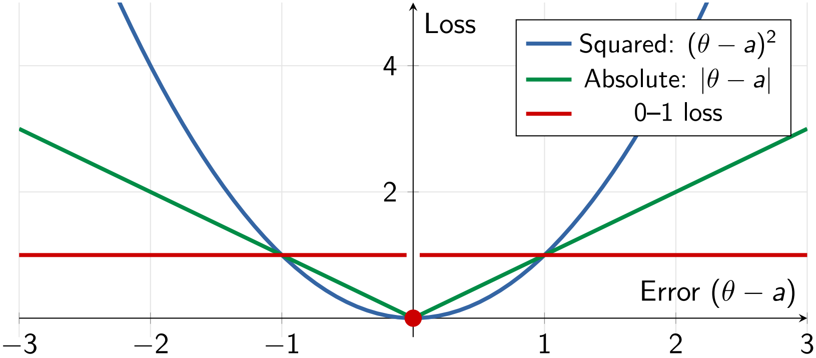
$L = \mathbf{1}[\theta \neq a]$

Wrong or right, nothing in between

$\Downarrow$

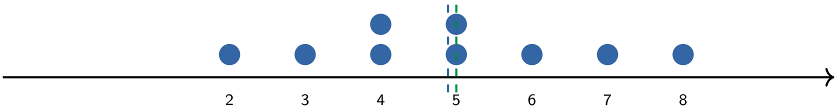
Mode

Visualizing the Losses

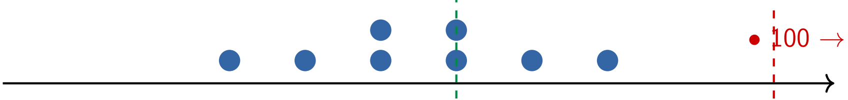


Mean vs Median: Sensitivity to Outliers

Dataset: $\{\mathbf{2, 3, 4, 4, 5, 5, 6, 7, 8}\}$



Now replace 8 with 100:



One outlier moved the mean from 4.9 to 15.1.
The median didn't budge.

The Mean Can Mislead

Three statisticians go hunting.

They spot a deer. The first one fires and misses **5 meters to the right**.
The second one fires and misses **5 meters to the left**.

The third one exclaims: *"We got him!"*

Average diet.

If one class of people eats **tup** and another eats **meat**,
then on average everyone eats **tolma**.

Risk and Empirical Risk

Risk (theoretical)

$$R(\theta, \hat{\theta}) = \mathbb{E}\big[L(\theta, \hat{\theta})\big]$$

Average loss over
all possible samples

(unknown — depends on F)

approximate $\dashrightarrow$

Empirical Risk

$$\hat{R} = \frac{1}{n}\sum_{i=1}^{n} L(X_i, a)$$

Average loss on the
data we actually have

(computable!)

Empirical Risk Minimization (ERM): choose the estimator that minimizes $\hat{R}$.
This principle unifies least squares, maximum likelihood, and most learning algorithms.

ERM in Machine Learning

Model $f_\theta(x)$ $\xrightarrow{\text{predict}}$ **Loss** $L(y_i, f_\theta(x_i))$ $\xrightarrow{\text{average}}$ **Minimize** $\hat{R} = \frac{1}{n} \sum L$

Linear regression Neural network Decision tree	MSE, MAE Cross-entropy Hinge loss	SGD Adam Closed-form

Every ML training procedure is ERM.

Choose a model class, choose a loss, minimize the empirical risk over parameters.

One Principle, Many Names

Statistics

Empirical Risk Minimization

$\min_\theta \frac{1}{n} \sum L(X_i, \theta)$

Machine Learning

Training = minimizing loss

$\min_\theta \frac{1}{n} \sum L(y_i, f_\theta(x_i))$

Find θ that best explains data

Likelihood Theory

Maximum Likelihood Estimation

$\max_\theta \sum \log f(X_i \mid \theta)$

Information Theory

Minimize KL divergence

$\min_\theta \; D_{\mathrm{KL}}(\hat{F}_n \,\|\, F_\theta)$

MLE with log-loss **=** **ERM** with neg. log-likelihood **=** **minimizing** KL divergence to data.

Questions?

Next lecture: Descriptive Statistics & Empirical Distributions