Covariance and Correlation

Hayk Aprikyan, Hayk Tarkhanyan

Covariance

In the table below, we have the number of customers of Ponchikanots who bought cacao and those who bought donuts, for 10 consecutive days.

Day	Cacao	Donuts
1	10	18
2	12	25
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6	11	22
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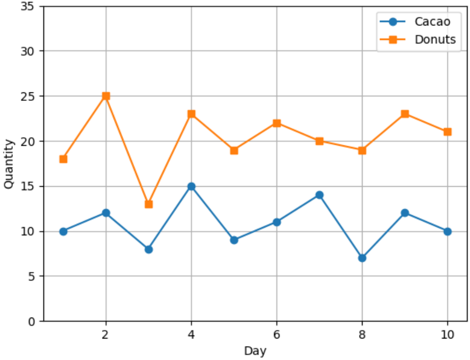

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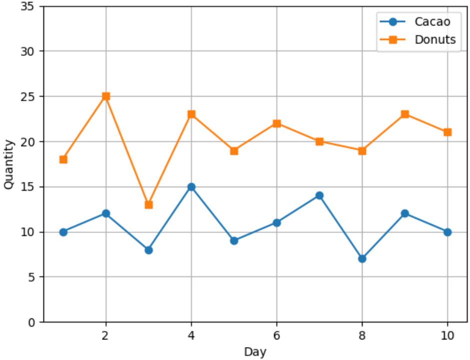

Does there seem to be any relation between the number of cacao and donuts sold?

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- When more cacao is sold, do more donuts get sold as well?

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3	$8 - 10.8$	$13 - 20.3$
4	$15 - 10.8$	$23 - 20.3$
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2	1.2	4.7	5.64
3	−2.8	−7.3	20.44
4	4.2	2.7	11.34
5	−1.8	−1.3	2.34
6	0.2	1.7	0.34
7	3.2	−0.3	−0.96
8	−3.8	−1.3	4.94
9	1.2	2.7	3.24
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- Take X and Y
- Subtract their means to see how much they deviate from the average

$$X - \mathbb{E}[X] \qquad Y - \mathbb{E}[Y]$$

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- Multiply those deviations
  (so that pos $\times$ pos $=$ pos, neg $\times$ neg $=$ pos, and pos $\times$ neg $=$ neg)

$$(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])$$

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This number is called the **covariance** between X and Y, and it measures how much the changes in X and Y are related to each other.

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The *covariance* between random variables X and Y is defined by

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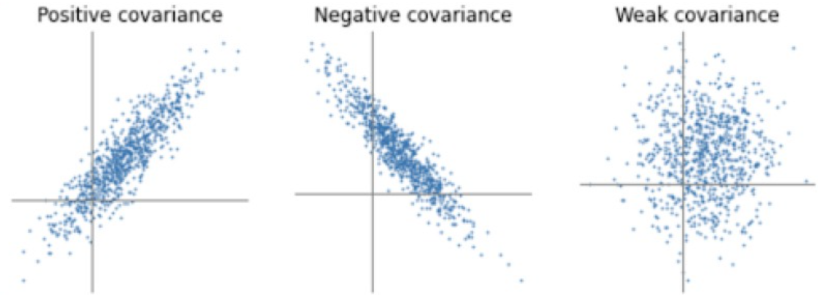
The most important part about covariance is its **sign.**

Covariance



Positive covariance
Negative covariance
Weak covariance

Covariance – alternative definition (optional)

Intuitively, we want covariance to measure the "dependence" between two random variables.

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- If Y is totally dependent on X, e.g. when $Y = 2X$, then:

$$\text{Var}[X+Y] = \text{Var}[X]+4\cdot\text{Var}[X]+2\cdot(\mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y])$$

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so the last term is pretty large.

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In other words, you have

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So the **difference** between

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But this difference is exactly equal to $\text{Cov}[X, Y]$ (times 2)!

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5. $\text{Cov}[X, X] = \text{Var}[X]$
6. And of course, if X and Y are independent, $\text{Cov}[X, Y] = 0$ (but the converse isn't true).

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Example

X	Y	with probability
0	3	$1/8$
1	2	$3/8$
2	1	$3/8$
3	0	$1/8$

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Covariance

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$X - \mathbb{E}[X]$	$Y - \mathbb{E}[Y]$	with probability
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The value of correlation is **always between** -1 **and** 1**.**

Correlation

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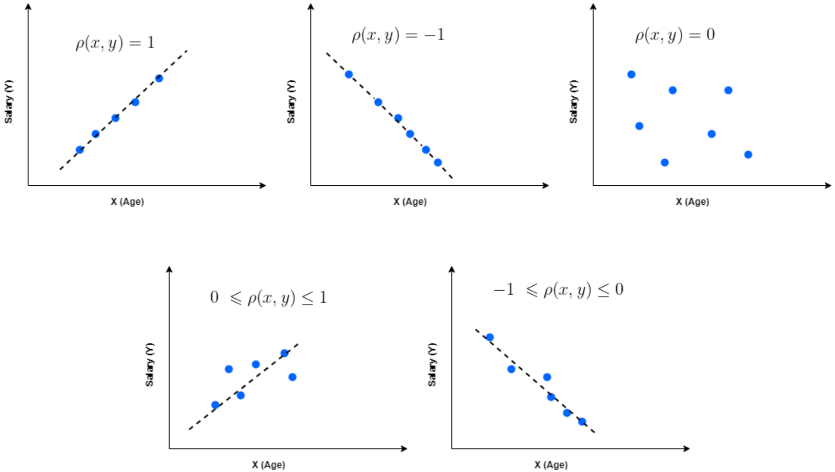
Whenever there is a **linear relationship** between X and Y:

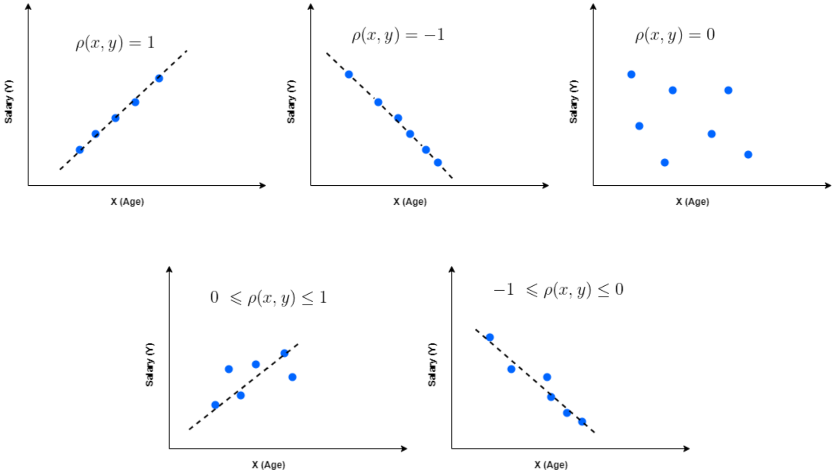
- $Y = X$
- or $Y = -X$
- or $Y = 2X + 3$

i.e. when $Y = aX + b$ for some constants a and b.

Correlation



Correlation



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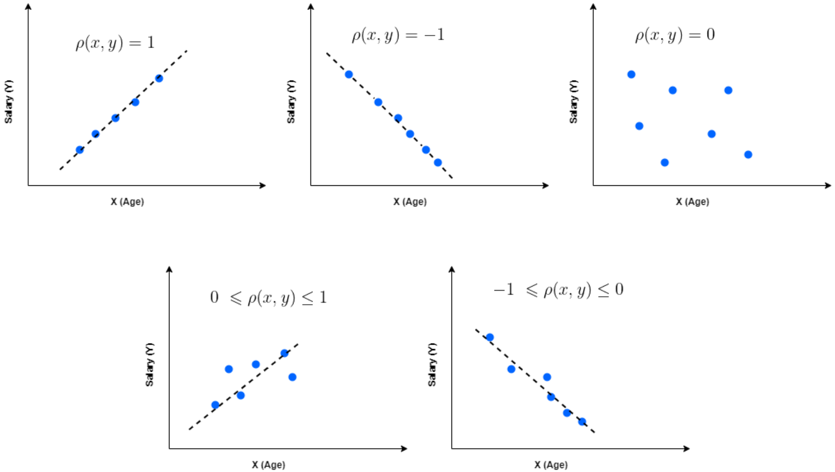

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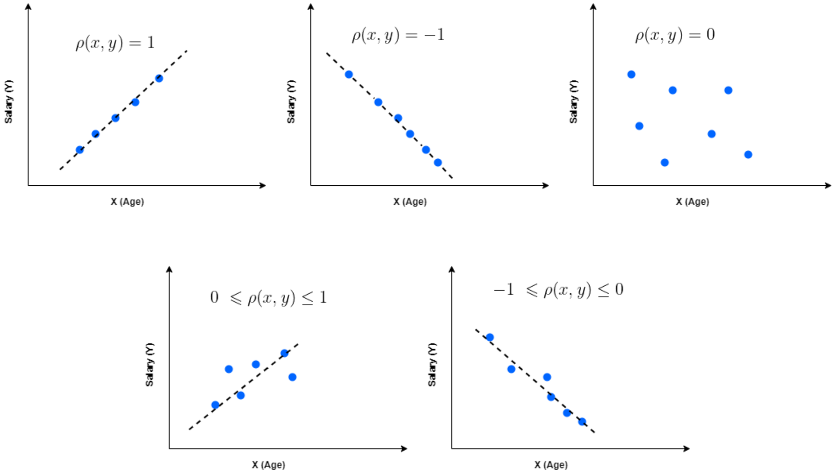

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In practice, instead of knowing the distributions of X and Y, we often have samples of their values, i.e. some data consisting of n rows and 2 columns:

X	Y
x_1	y_1
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Then the sample correlation between X and Y is computed as:

$$r_{X,Y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where $\bar{x}$ and $\bar{y}$ are the sample means of X and Y.

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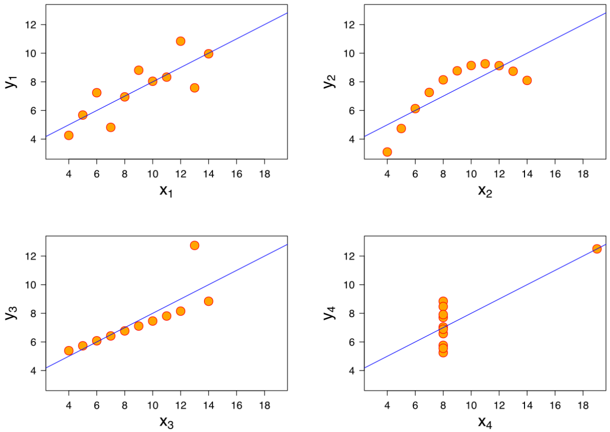

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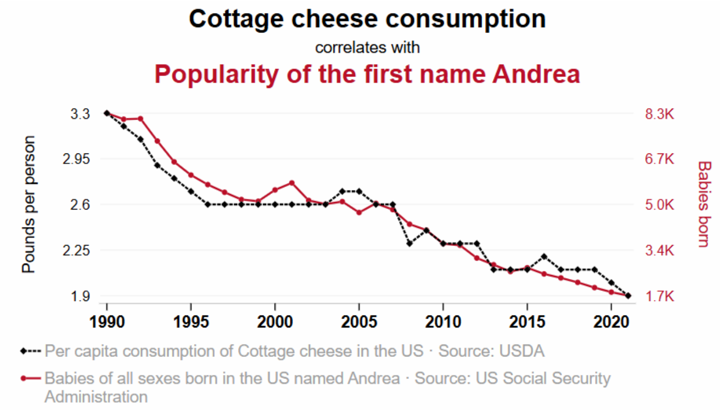

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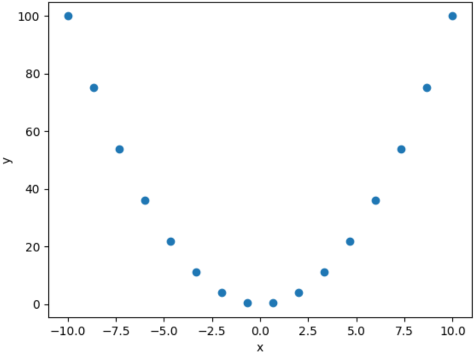

Here $Y = X^2$.
Yet, the correlation between X and Y is just 0!

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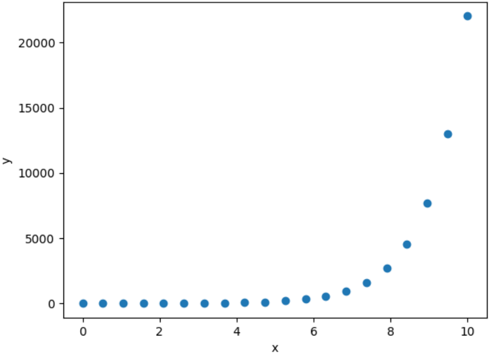

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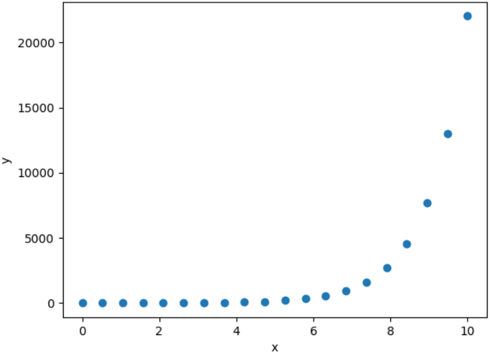

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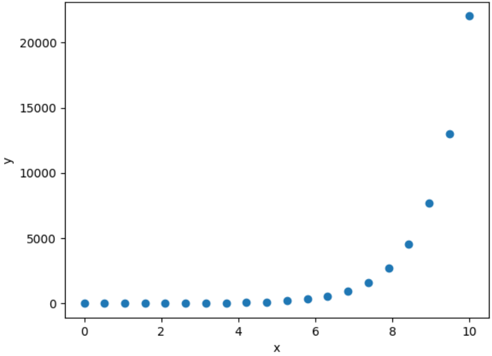

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Can we measure dependence between random variables in another way?

Spearman's Rank Correlation Coefficient

One way to measure whether

$$X \text{ increases} \quad \Rightarrow \quad Y \text{ increases}$$

is to rank the values of X and Y:

1. sort the values of X in increasing order and assign ranks to them (the smallest value gets rank 1, the second smallest gets rank 2, etc.)
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This is called *Spearman's rank correlation coefficient*, and it measures the **monotonic** relationship between two random variables, i.e. whether one variable tends to increase when the other increases.

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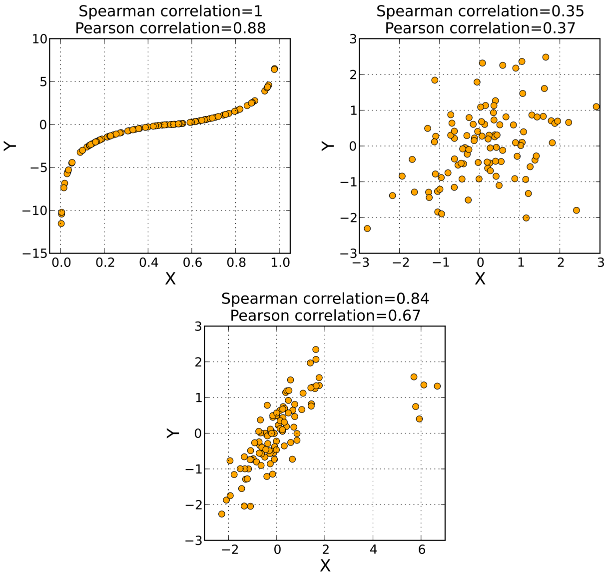

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- **Triangle Inequality:** $\sqrt{\text{Var}[X + Y]} \leq \sqrt{\text{Var}[X]} + \sqrt{\text{Var}[Y]}$