Expected Value, Variance

Hayk Aprikyan, Hayk Tarkhanyan

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Would you play this game? What if instead of \$36, you won \$150 if it fell on 8?

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Since the chance of winning is only $\frac{1}{38}$, if you play it a couple of thousands times (say 38000), then you can expect to win about ≈ 1000 times and lose ≈ 37000 times. Your net revenue would then be:

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Should we say the average winning is $\frac{1000000+(-300)}{2} = 499.850$ dram? No! The chances of winning are 3999 times less than the chances of losing:

$$\mathbb{P}[X = 1.000.000] = \frac{1}{4000} < \frac{3999}{4000} = \mathbb{P}[X = -300]$$

and we should take this into account.

Expected Value

The actual average value of a random variable X is called the *expected value* or the *expectation* of X and is denoted by $\mathbb{E}[X]$.

Definition

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In words, the expected value is the **weighted average** of all its possible values – where each of the values is weighted by its probability.

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In the lottery example, the expected winning amount is:

$$\mathbb{E}[X] = 1000000 \cdot \frac{1}{4000} + (-300) \cdot \frac{3999}{4000} = -50.25$$

i.e. on average, you would be losing 50.25 dram per ticket.

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$$\mathbb{E}[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

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It is important to note that the expected value **does not have to be** one of the possible values of the random variable! In the above example, X can only take integer values from 1 to 6, yet its expected value is 3.5.

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Could be also ask about the square of the distance from 0, i.e. what is $\mathbb{E}[X^2]$?

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Theorem

If X is a discrete random variable, then

$$\mathbb{E}[g(X)] = \sum_{x} g(x) \cdot \mathbb{P}[X = x]$$

If X is a continuous random variable, then

$$\mathbb{E}[g(X)] = \int_{a}^{b} g(x) \cdot f_X(x)\, dx$$

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Note that $\mathbb{E}[X^2] \neq (\mathbb{E}[X])^2$, and in general, $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$!

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Question

If $\mathbb{E}[X] = 5$, what do you think is $\mathbb{E}[2X + 3]$?

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Theorem

If X and Y are independent random variables, then

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

The converse is not always true.

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If X denotes the winnings of the first game, and Y of the second game, we can say that Y has a **higher variance** than X:

$$Y - \mathbb{E}[Y] \text{ on average} \quad > \quad X - \mathbb{E}[X] \text{ on average}$$

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We need to specify what "on average" means here.

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The standard deviation shows how much, *on average*, do the values of the random variable deviate from their average $\mathbb{E}[X]$.

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and the standard deviation is:

$$\sigma_X = \sqrt{\text{Var}[X]} \approx \sqrt{2.92} \approx 1.71$$

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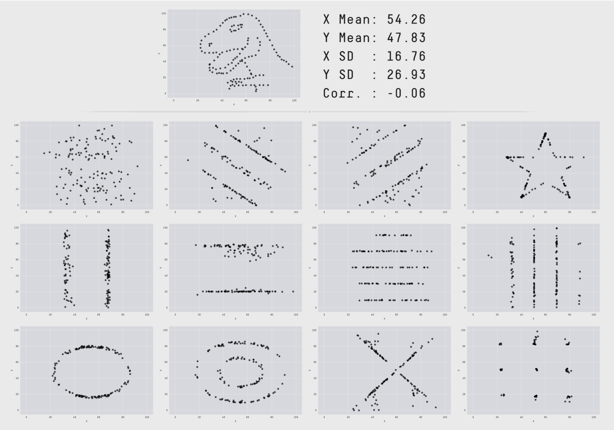
Why do you think the 4th point makes sense?

Variance

Warning

Expected value and variance are very useful to describe random variables, **but they are not everything!** They do not replace CDF/PDF/PMF!



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Your friend who on average earns 5000 dram per day, boasts that each month, there are at least 8 days when he manages to earn more than 20000 dram. Is that possible?

Suppose X is any random variable (discrete or continuous) and a is any positive number.

Markov's inequality

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

Applying this to our example, we get:

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so your friend is lying!

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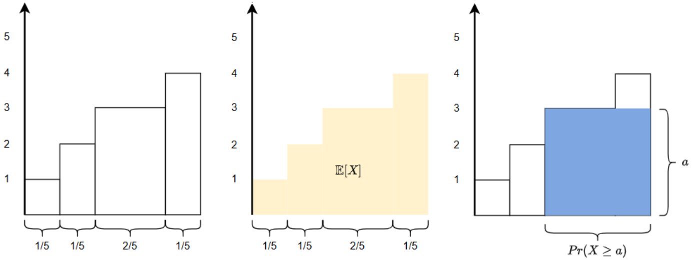
We can also prove Markov visually. Let X be a random variable taking values $\{1, 2, 3, 4\}$ with probabilities:

$$\mathbb{P}[X = 1] = 0.2, \quad \mathbb{P}[X = 2] = 0.2, \quad \mathbb{P}[X = 3] = 0.4, \quad \mathbb{P}[X = 4] = 0.2$$



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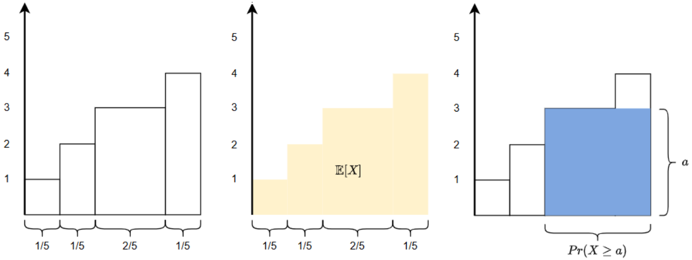

Question

Can you use Markov to prove *Chebyshev's inequality*?

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}$$

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Hint: Use that $\text{Var}[X] \geq 0$.

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Why is this true? Because we are plugging in X into the function $g(x) = x^2$, which is a **convex** function.

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In analysis, we had this Jensen's inequality:

$$\alpha_1 g(x_1) + \cdots + \alpha_n g(x_n) \geq g\left(\alpha_1 x_1 + \cdots + \alpha_n x_n\right)$$

where $\alpha_1 + \cdots + \alpha_n = 1$, i.e.

$$\text{average of } g(x) \geq g(\text{average of } x)$$

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Same thing holds for random variables:

Jensen's Inequality

If X is a random variable and $g(x)$ is any convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$$

Cauchy-Schwarz Inequality (optional)

In **linear algebra**, we had this Cauchy-Schwarz inequality for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$:

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In particular, if $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = 0$, this becomes:

$$\left|\mathbb{E}[XY]\right| \leq \sqrt{\text{Var}[X]} \cdot \sqrt{\text{Var}[Y]}$$

Sample Mean and Sample Variance (optional)

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Let X denote the height of a randomly chosen person from Artik. How would you estimate $\mathbb{E}[X]$ and $\text{Var}[X]$?

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Definition

The average of the samples is called the *sample mean*:

$$\overline{x} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

and the quantity below is called the *sample variance*:

$$s^2 = \frac{(X_1 - \overline{x})^2 + (X_2 - \overline{x})^2 + \cdots + (X_n - \overline{x})^2}{n - 1}$$