Random Variables

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Random Variables

Recap:
During the previous lecture we had a problem like this:

Example

Both the bus and you get to the bus stop at random times between 12 pm and 1 pm. When the bus arrives, it waits for 5 minutes before leaving. When you arrive, you wait for 20 minutes before leaving if the bus doesn't come. What is the probability that you catch the bus?

Here we did not know the **exact values** of the arrival times, so in order to compare them, we **denoted them** by y and b – and calculated their probabilities.

In this case, we say that y and b are **random variables**.

Random Variables

In general, we often deal with situations where some quantity is unknown:

- i.e. we don't know its exact value,

but we know the **probabilistic distribution** of its values, i.e.

- what possible values it can take, and
- how probable each value is.

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Formally, any unknown X for which we can find the probability that

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is a random variable. In other words, any quantity X about which we can answer questions like *"What is the probability that X is less than or equal to k?"* or *"What is the probability that X equals k?"* is a random variable.

Random Variables (optional)

Technically speaking, this means there should be a way to measure the probabilities

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Should we worry about this technicality? Fortunately, no:
In practice, *every unknown thing* is a random variable.

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Two fair dice are thrown.

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are **not** random variables.

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We throw a fair die and win \$1 if it is prime, lose \$1 if it is composite, and stay even otherwise. If we denote the outcome of the die by ω,

$$X = \begin{cases} 1, & \text{if } \omega \in \{2, 3, 5\} \\ -1, & \text{if } \omega \in \{4, 6\} \\ 0, & \text{if } \omega \in \{1\} \end{cases}$$

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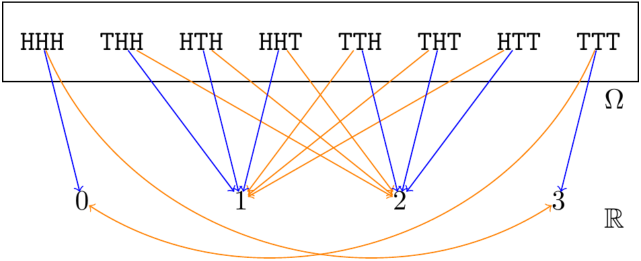
Example

The number of letters in a randomly selected word is a random variable.

PMF

Suppose we toss a fair coin 3 times, and we are interested in:

- X = the number of heads
- Y = the number of tails



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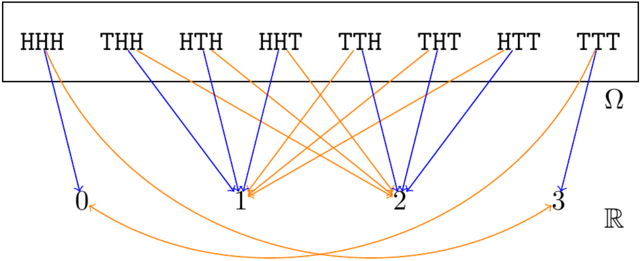

Question

What are the possible values of X and Y, with their probabilities?

PMF

The possible values of X are $0, 1, 2, 3$:

$$X = \begin{cases} 0 & \text{if } \omega \in \{\mathtt{TTT}\} \\ 1 & \text{if } \omega \in \{\mathtt{HTT}, \mathtt{THT}, \mathtt{TTH}\} \\ 2 & \text{if } \omega \in \{\mathtt{HHT}, \mathtt{THH}, \mathtt{HTH}\} \\ 3 & \text{if } \omega \in \{\mathtt{HHH}\} \end{cases}$$

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A convenient way to represent this information:

$$\text{"}\textit{What is the probability that } X = k\text{?"} \quad \text{i.e.} \quad \mathbb{P}[X = k]$$

is to denote it as:

$$p_X(k)$$

and call it the PMF of X.

Definition

The function which takes k and returns the probability that $X = k$:

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If you know the PMF of X, you know everything about the *distribution* of X (i.e. how likely each value is).

PMF

In our example, the PMF of X is:

k	$p_X(k)$
0	$1/8$
1	$3/8$
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The sum of all probabilities in the PMF is 1.

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A pitiful student knows 4 out of 15 topics, and there will be 2 questions on the exam.

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$$\mathbb{P}[X = 0] = \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{21} \qquad \text{(rip student)}$$

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PMF

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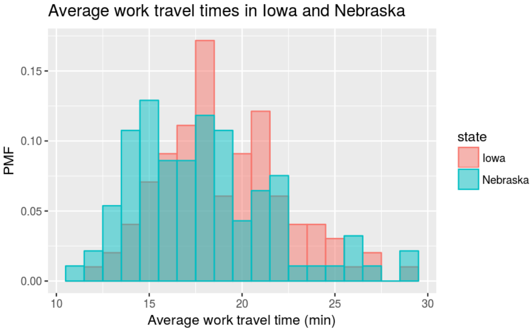
k (right answers)	$p_X(k)$
0	0.52
1	0.42
2	0.06

CDF

Let's look at the PMFs of X and Y, showing average travel times (in minutes) for two US states:



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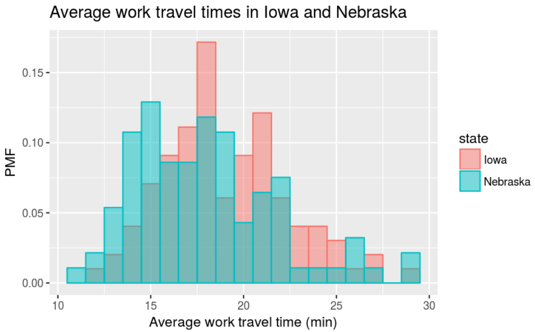

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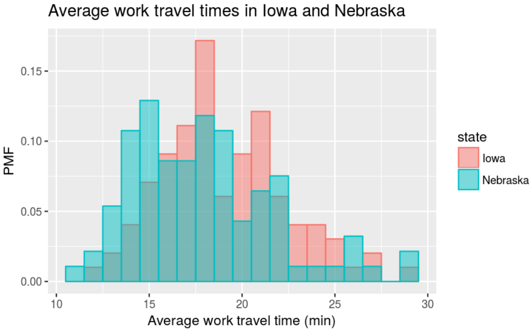

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$$F_X(10) = \mathbb{P}[X \leq 10] = ?$$

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The function that takes k and returns the probability that X is less than or equal to k:

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Properties

- For any $x \in \mathbb{R}$, $0 \leq F_X(x) \leq 1$
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- $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$

CDF

Example

Let X indicate the number on a fair die. Then

$$F_X(6.1) =$$

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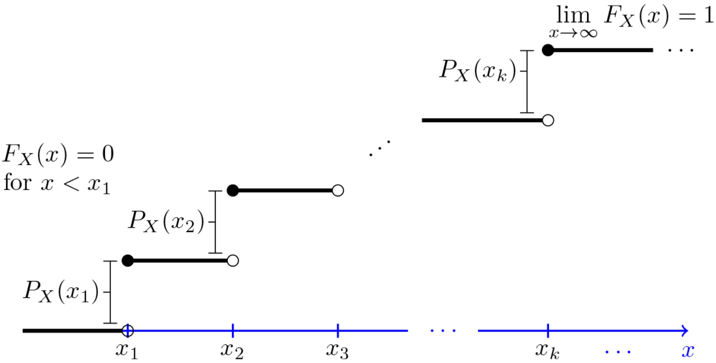
$$F_X(x) = \begin{cases} 0, & \text{if } x < 1, \\ \frac{1}{6}, & \text{if } 1 \leq x < 2, \\ \frac{2}{6}, & \text{if } 2 \leq x < 3, \\ \frac{3}{6}, & \text{if } 3 \leq x < 4, \\ \frac{4}{6}, & \text{if } 4 \leq x < 5, \\ \frac{5}{6}, & \text{if } 5 \leq x < 6, \\ 1, & \text{if } x \geq 6 \end{cases}$$

CDF

Graphs of CDFs usually usually look like this:



Identically distributed RVs

Suppose we have a second pitiful student who has also studied for 4 topics out of 15. Let $Y =$ number of questions the second student gets correct.

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- The students may be asked different questions and get different grades, so $X \neq Y$.
- However, Y has the same PMF and CDF as X (check it!)

In this case, we say that X and Y are identically distributed:

Definition

Two random variables are *identically distributed* if their PMFs/CDFs are equal.

Identically distributed RVs

In some sense, identically distributed random variables are "similar" to each other, **but not necessarily** equal.

Example

Say we toss a fair coin. X and Y are RVs such that:

$$X = \begin{cases} 0, & \text{if } \omega = \mathtt{H}, \\ 1, & \text{if } \omega = \mathtt{T}, \end{cases} \qquad Y = \begin{cases} 1, & \text{if } \omega = \mathtt{H}, \\ 0, & \text{if } \omega = \mathtt{T}, \end{cases}$$

They both have the save PMFs:

$$\mathbb{P}[X = 0] = \mathbb{P}[X = 1] = \mathbb{P}[Y = 0] = \mathbb{P}[Y = 1] = \frac{1}{2}$$

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Random Variables

So far we have only seen random variables that take a **finite** or **countable** number of values, like $\{2, 3, 4, \dots, 12\}$ or $\{0, 1, 2, \dots\}$ – because they showed some quantities like the number of heads, number of goals, etc.

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Otherwise,

Definition

If the values of X cannot be represented as a list (e.g. they are an interval), it is called a *continuous random variable.*

Random Variables

Example

Examples of discrete random variables:

- the sum of two consecutive dice rolls,
- the number of goals in a football match,
- the difference between the scores of two teams,
- the number of children in a randomly selected family,
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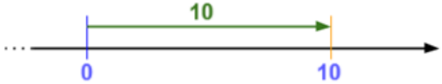
Example

Examples of continuous random variables:

- the time until a bus arrives,
- the height of a randomly selected person,
- the stocks of Apple tomorrow,
- the x-coordinate of a randomly selected point on the (x, y)-plane.

PDF

Okay, what if X is a continuous random variable?



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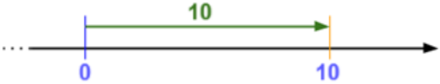

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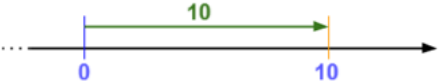

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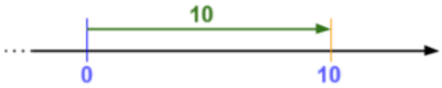

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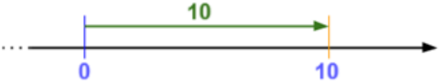

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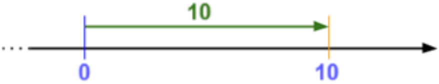

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So another question we might ask is:

Question

What do you think $\mathbb{P}[X \leq 5]$ is?

PDF

While for discrete random variables both PMF and CDF make sense, for continuous random variables PMF is useless, but still can rely on CDF:

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If X is a continuous random variable, then there exists a function $f(x) \geq 0$ such that for any $c \in \mathbb{R}$,

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The function f is called the *probability density function* or *PDF* of X.

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The PDF of a continuous random variable plays essentially the same role as the PMF of a discrete random variable.

PDF

Just like the density of an object measures the concentration of mass (per unit volume), the probability density function captures the density of *probability* at point x:

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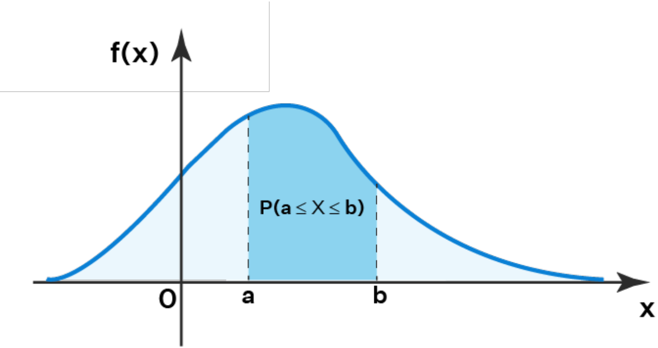

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5. $\mathbb{P}[X = a] = \mathbb{P}[X \leq a] - \mathbb{P}[X < a]$

PDF

Example

Ani chooses a random real number X uniformly from the interval $[a, b]$.

By "uniformly" we mean that for any two intervals of the same length (e.g. $(1.3, 1.5)$ and $(4.7, 4.9)$) X can belong to them with the same probability.

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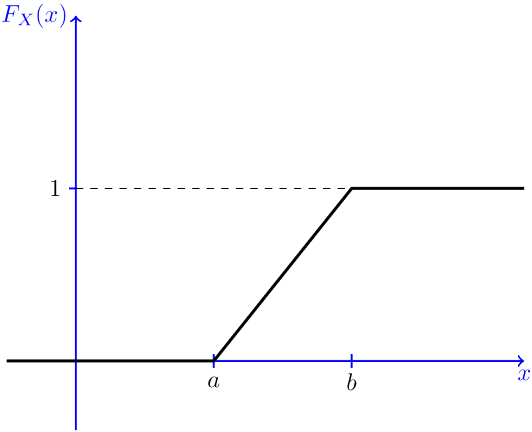
For $a \leq x \leq b$, we have:

$$F_X(x) = \mathbb{P}[X \leq x] = \mathbb{P}[X \in [a, x]] = \frac{x - a}{b - a}$$

PDF



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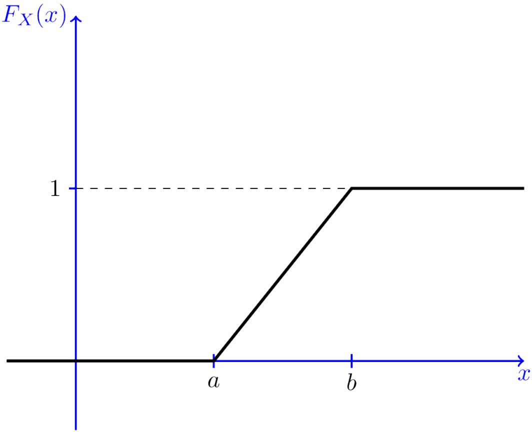

Thus,

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

PDF

To find $f_X(x)$, we take the derivative of $F_X(x)$:

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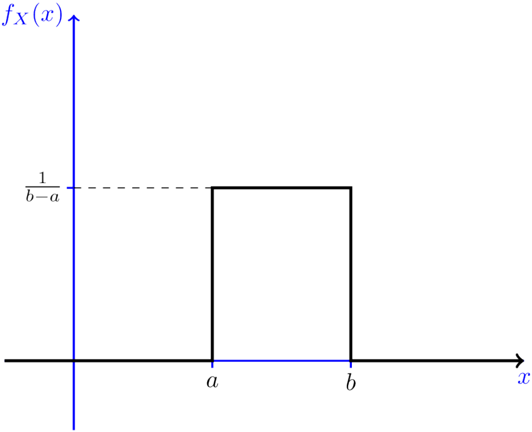

Summary

To summarize, all random variables come in two types:

	PMF	PDF	CDF
discrete	✓	–	✓
continuous	–	✓	✓

Independence

Recall that two events A and B are called independent if they do not affect each other, i.e.

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Definition

X and Y are called *independent* if

$$\mathbb{P}[X \leq a \text{ and } Y \leq b] = \mathbb{P}[X \leq a] \cdot \mathbb{P}[Y \leq b]$$

for any $a, b \in \mathbb{R}$.

So the probability of both X and Y simultaneously being less than some numbers is just their *separate* probabilities multiplied together.