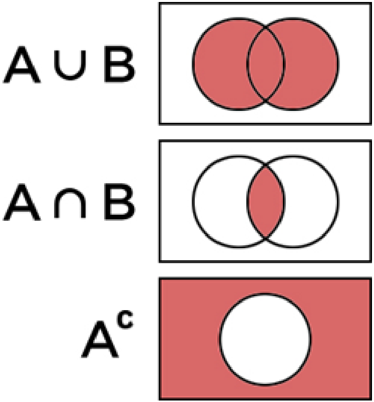
Probability, Independence, Bayes Rule

Hayk Aprikyan, Hayk Tarkhanyan

Preliminaries

Recall the set operations:



Union $A \cup B$:
All elements that belong to A or B or both

Intersection $A \cap B$:
All elements that belong to *both* A and B

Complement A^c:
All elements that do not belong to A

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Questions like these might not have precise answers – as long as you haven't rolled the dice or tossed the coin yet, the actual outcomes are unknown.

But even though we cannot know the exact outcomes in advance, we can still try to estimate their **probabilities**, i.e. *how likely* different outcomes are to happen.

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The two possibilities (**Heads** and **Tails**) are called *outcomes* and the set of all outcomes is called *sample space:*

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First, we should figure out how to mathematically represent the question.

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We bring together all the even numbers from the sample space:

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- How do you denote the probability that the outcome is less than 4 **and also** even?
- How do you denote the probability that the outcome is less than 4 **or** even?

Examples

Example

Playing chess with Levon Aronian is a random experiment, where:

- Outcome is the result of the game.
- Possible outcomes are: **Win**, **Lose**, and **Draw**.
- The sample space is:

$$\Omega = \{\mathbf{Win}, \mathbf{Lose}, \mathbf{Draw}\} = \{W, L, D\}$$

- But the probabilities of these outcomes are not equal :)

Example

A football match is a random experiment, where:

- Outcome is the score of the game.
- For example, one possible outcome is "Pyunik 2 - 1 Alashkert". One way to denote it is $(2, 1)$, so:

$$\Omega = \{(0,0); (0,1); (1,0); (1,1); (2,0); \dots\}$$

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- Outcome is the time until the bus arrives.
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Whenever Ω is an interval (e.g. $(0, 1), [1, 10], [0, +\infty)$), the probability of each outcome is 0.

(of course, this does not mean that they are impossible – watch this!)

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So in this case, we can define our event space $\mathcal{F}$ to only include the set of all Armenian speakers, and its complement (non-Armenian speakers).

Example

In another experiment (e.g. when playing backgammon) we may be interested in the event of our number being 2 or not. In that case, we will consider the following event space:

$$\mathcal{F} = \{\varnothing, \{2\}, \{1, 3, 4, 5, 6\}, \Omega\}$$

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Two events A and B of the same experiment are called *disjoint* or *mutually exclusive* if $A \cap B = \varnothing$.

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When rolling a die, the events $A = \{1, 4\}$ and $B = \{2, 5\}$ are disjoint, while A and $C = \{3, 4, 5\}$ are not.

Example

When waiting for a bus, the events $A = [0, 20]$ and $B = [30, 40]$ are disjoint, but none of them is disjoint with $C = [10, 40]$.

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Experiments like this (where all outcomes have the same probability of occuring) are called *equiprobable*.

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since A has 3 elements and the total number of outcomes is 6.

In equiprobable case, the probability of any event $A \in \mathcal{F}$ is:

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Now we roll two fair dice. Our sample space will be
$\Omega = \{(x, y) \mid 1 \leq x, y \leq 6\}$.
Assuming that all of 36 outcomes are equally likely to show up, the probability of each (x, y) outcome is:

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$$\mathbb{P}\big[(x, y)\big] = \frac{1}{36} \qquad \text{for any } (x, y) \in \Omega$$

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As it shows, probability can also be not equiprobable.

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So in general, in different problems the *probability measure* $\mathbb{P}$ can be different, and it depends on the specifics of the problem how it is computed.

There are, however, three properties which the probability measure $\mathbb{P}$ always satisfies:

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2. $\mathbb{P}[\Omega] = 1$,
3. For any **disjoint** events $A_1, A_2, \ldots,$

$$\mathbb{P}\left[A_1 \cup A_2 \cup \ldots\right] = \mathbb{P}[A_1] + \mathbb{P}[A_2] + \ldots$$

i.e.

$$\mathbb{P}\left[\bigcup_n A_n\right] = \sum_n \mathbb{P}[A_n]$$

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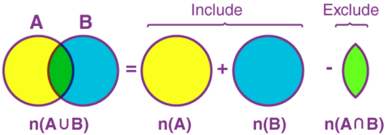

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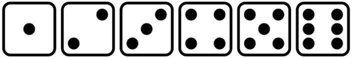

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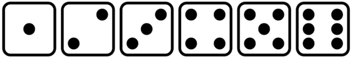

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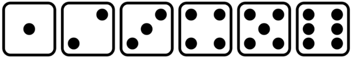

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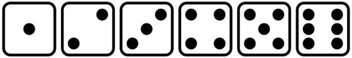

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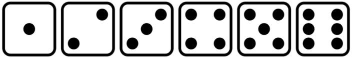

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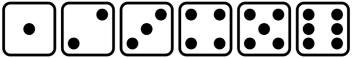

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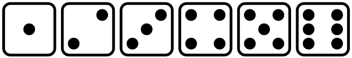

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Here we know that the outcome **cannot be** 1, 4 or 6. Therefore we are left with only three possible outcomes $B = \{2, 3, 5\}$, out of which only 2 is even. So in this case the probability of being even is $\frac{1}{3}$.

Conditional Probability

Definition

For any events A and B (if $\mathbb{P}[B] \neq 0$), the following number:

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

is called the **conditional probability** of A **given** B
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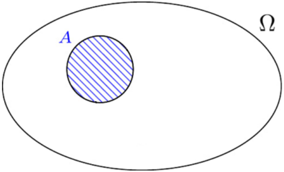

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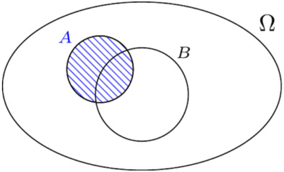

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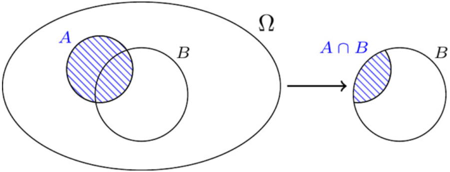

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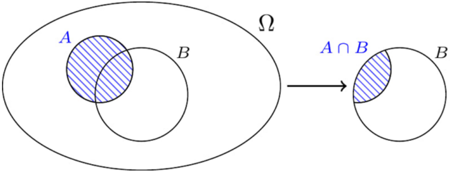

In our problem, if A means being even and B means being prime, we had:

$$\mathbb{P}[\text{even} \mid \text{given that prime}] = \mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{1/6}{3/6} = \frac{1}{3}$$

Conditional Probability

Example

Suppose we roll two fair dice. What is the probability that the first one is 2, given that their sum is no greater than 5?

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	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Usually, there would be 36 total outcomes, but since we *know* the sum is 5, they are only 10 possible outcomes left. Out of them only three outcomes ("2-1", "2-2" and "2-3") are desired. So the probability is $\frac{3}{10}$.

Law of Total Probability

Question

In some university, 3/5 of all students are women and the rest are men. It is known that 15% of men are over left-handed, while only 10% of women are. If you choose a random student, what is the probability of the student being left-handed?

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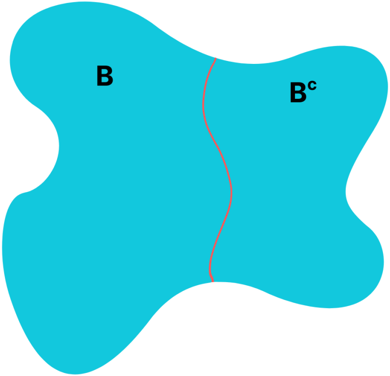
Let B denote the event that the randomly selected student is a woman, and B^c that he is a man.

With another letter, say A, let us denote the event of being left-handed.

Law of Total Probability



$\mathbb{P}[A] =$

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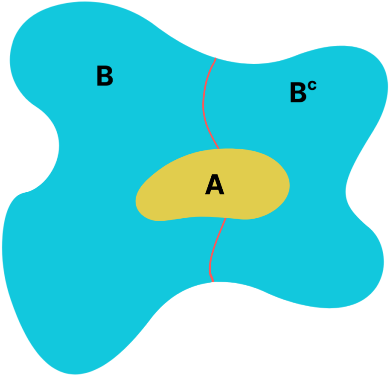

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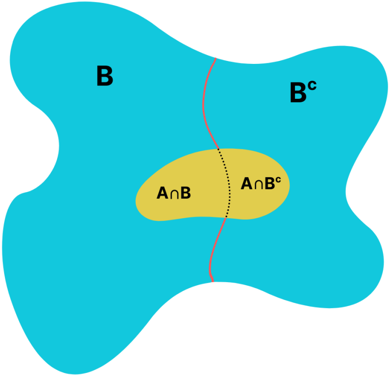

$$\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^c] =$$

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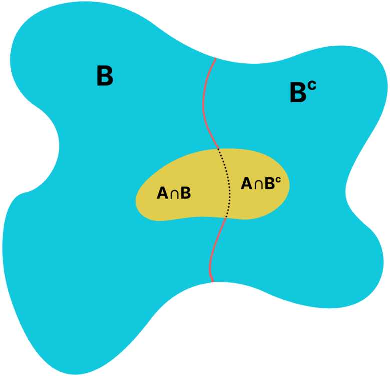


$$\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^c] = \mathbb{P}[B] \cdot \mathbb{P}[A|B] + \mathbb{P}[B^c] \cdot \mathbb{P}[A|B^c]$$

Law of Total Probability

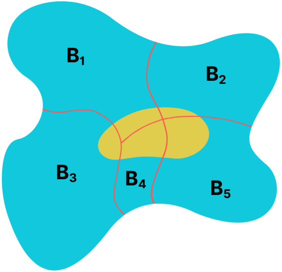
What we get is known as the Law of Total Probability:

Theorem

If A and B are some events such that $\mathbb{P}(B) \neq 0$, then

$$\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c)$$

Law of Total Probability



We can also generalize it to the case of three or more subgroups:

Theorem

If $B_1, B_2, \ldots, B_n$ are some disjoint events such that $A \subset \bigcup_{k=1}^{n} B_k$, then

$$\mathbb{P}(A) = \mathbb{P}(B_1) \cdot \mathbb{P}(A|B_1) + \cdots + \mathbb{P}(B_n) \cdot \mathbb{P}(A|B_n)$$

Law of Total Probability

Example

There are 52 cards in a deck. One of the cards is randomly removed from the deck, 51 are left. What is the probability that if we take a card, it will be a diamond?

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If we denote by A the probability of taking a diamond, and by B the probability that the removed card was a diamond, then either:

- the taken card was a diamond, and there are 12 left,
- the taken card wasn't a diamond, and there are 13 left.

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The total probability will be:

$$\mathbb{P}[A] = \mathbb{P}[B] \cdot \mathbb{P}[A|B] + \mathbb{P}[B^c] \cdot \mathbb{P}[A|B^c]$$

$$\mathbb{P}[A] = \frac{13}{52} \cdot \frac{12}{51} + \frac{39}{52} \cdot \frac{13}{51} = \frac{1}{4}$$

Bayes Rule

One more simple yet powerful tool is the so called Bayes Rule:

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If A and B are some events (with non-zero probabilities), then

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A|B] \cdot \mathbb{P}[B]}{\mathbb{P}[A]}$$

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Let F denote dangerous fire, S smoke. Then:

$$\mathbb{P}[F|S] = \frac{\mathbb{P}[F] \cdot \mathbb{P}[S|F]}{\mathbb{P}[S]} = \frac{1}{100} \cdot \frac{90}{100} : \frac{10}{100} = 0.09$$

Independence

Now let's consider another simple experiment: flipping a fair coin and rolling a six-sided die. The probability of getting **Heads** on the coin flip is independent of the probability of rolling a specific number on the die, as these two events *do not affect* each other.

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This definition makes sense because it means that the probability of both A and B occurring together is simply the product of their individual probabilities: the outcome of one event has no effect on the other.

Independence

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
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Suppose we are rolling two fair dice (for the last time).
Event $A =$ the first die is even, $B =$ the second die is odd.

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$$\mathbb{P}[A \cap B] = \frac{9}{36} = \frac{1}{4}$$

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3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
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When solving a problem, principles like total probability, Bayes rule, and independence help us a lot. Let's explore one more useful problem solving technique.

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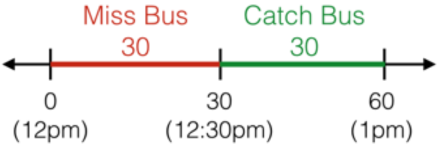


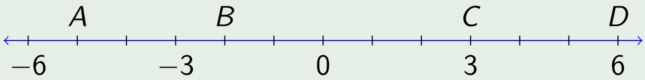
If we draw the hours on a number line and mark the time points for which we will miss or catch the bus, we see that the *length* of the segment for catching the bus is half the total length:

$$\mathbb{P}[\text{catching the bus}] = \frac{30}{30 + 30} = \frac{1}{2}$$

Geometric Probability in 1D

Example

A selection is to be made between points A and D as seen below



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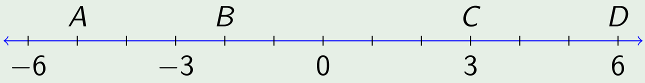

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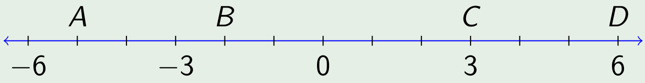

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So in general, we draw the sample space and the set of desired outcomes as *lines* or *line segments,* and divide

$$\frac{\text{length of desired outcomes}}{\text{length of all outcomes}}$$

to get the probability.

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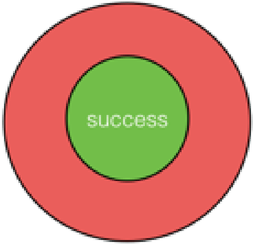
Imagine now that we are playing darts with two circles, the smaller circle has radius r, and the bigger circle has radius $2r$. A dart is thrown at random.



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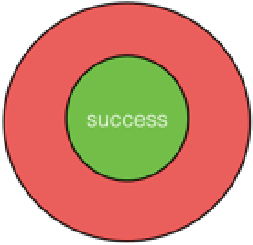

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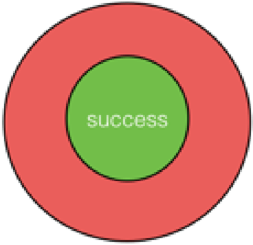

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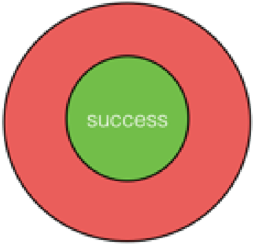

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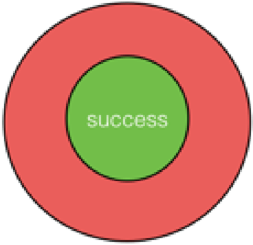

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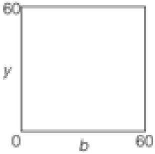

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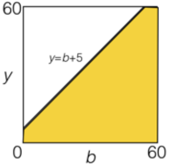
Now we need to find the region of successful outcomes, i.e. the points where we catch the bus. Since the bus will wait for 5 minutes, you need to arrive within 5 minutes of the bus' arrival, so $y \leq b + 5$.



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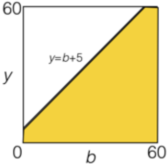
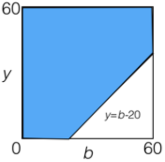


However, you only wait for 20 minutes, so you can't arrive more than 20 minutes before the bus, so $y \geq b - 20$.



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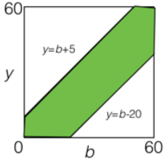
Combining our two conditions, the successful outcomes are:



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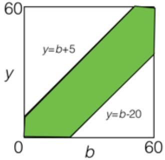

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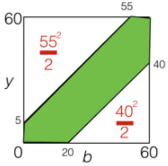

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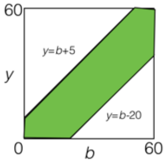

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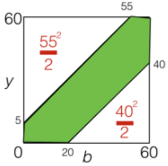

$$\mathbb{P}[\text{catching the bus}] = \frac{60^2 - \frac{55^2}{2} - \frac{40^2}{2}}{60^2} = \frac{103}{288}$$

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