Taylor Series, Integral

Hayk Aprikyan, Hayk Tarkhanyan

Convex and Concave Functions

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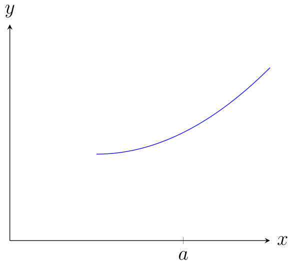

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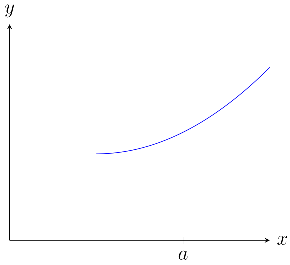

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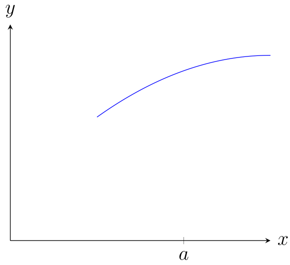

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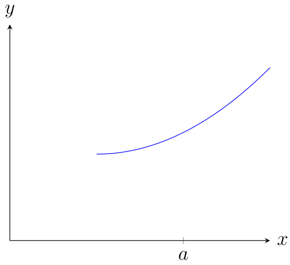

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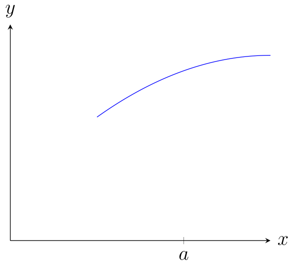
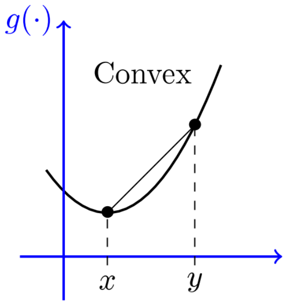
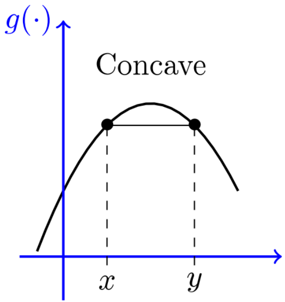


How can you determine which way it is?

Convex and Concave Functions





Definition

We say that $f(x)$ is *convex* on some interval if for any two points on its graph, the line connecting them always lies **above** the graph of $f(x)$.

Definition

We say that $f(x)$ is *concave* on some interval if for any two points on its graph, the line connecting them always lies **below** the graph of $f(x)$.

Convex and Concave Functions

More technically, a function is

- convex if
$$f(\alpha p + (1-\alpha)q) \leq \alpha f(p) + (1-\alpha) f(q)$$
- concave if
$$f(\alpha p + (1-\alpha)q) \geq \alpha f(p) + (1-\alpha) f(q)$$

for any two points p and q, and for any number $0 \leq \alpha \leq 1$.

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We will not prove this but open the link and play with α to get a feeling of what this means:

- www.desmos.com/calculator/ujoh0mh59d

Convex and Concave Functions

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Convex and Concave Functions

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Again, derivative is all you need!

Theorem

If $f(x)$ is twice-differentiable (i.e. there exists $f''(x)$), then:

- f is convex if and only if $f''(x) \geq 0$
- f is concave if and only if $f''(x) \leq 0$

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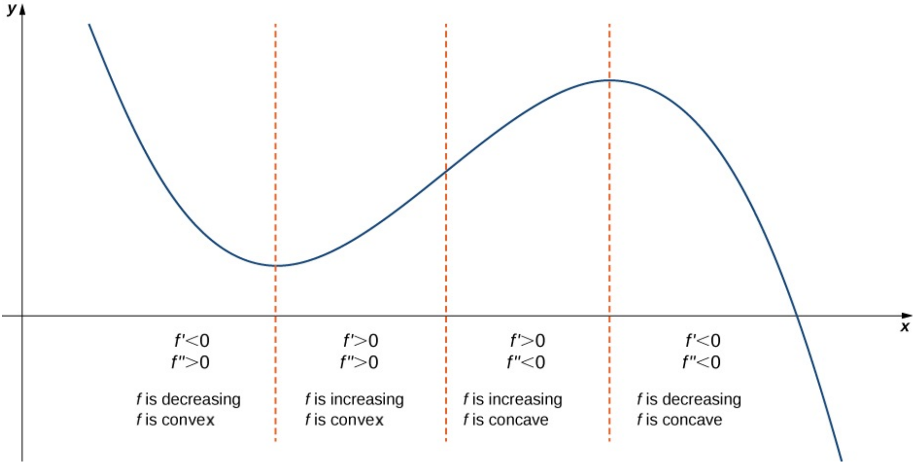
Examples

1. $f(x) = x^2$ is convex on $(-\infty, \infty)$.
   $f''(x) = 2 > 0$
2. $f(x) = -x^2$ is concave on $(-\infty, \infty)$.
   $f''(x) = -2 < 0$
3. $f(x) = x$ is both convex and concave on $(-\infty, \infty)$.
   $f''(x) = 0$

Convex and Concave Functions



Taylor Series

As we saw, the derivatives of a function tell so much about them. Can they be used to approximate the function?

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Suppose we have a function $f(x)$. Let us construct another function $g(x)$ which, at some point a has (or, equivalently):

- $g(a) = f(a)$
- $g'(a) = f'(a)$
- $g''(a) = f''(a)$
- $\dots$

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For example, suppose $f(x) = e^x$. Then $f(0) = 1$ and $f'(0) = 1$.

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$$P_k(x) = f(a)$$

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If we take $k = \infty$, we will get an infinite sum which is called the *Taylor series* of $f(x)$ around the point a.

Taylor Series

Taylor's Theorem

If a function $f(x)$ is k times differentiable at the point a, then

$$P_k(x) \to f(x), \qquad k \to \infty$$

around some (maybe small) interval around a.

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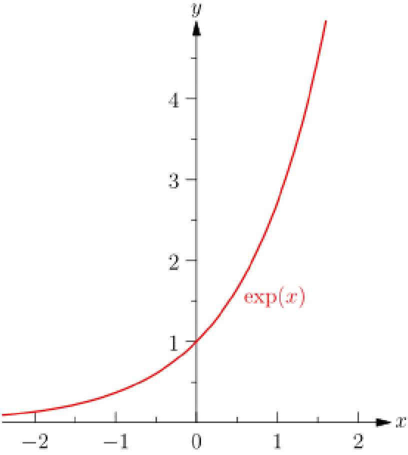
Example

- The Taylor series for any polynomial is the polynomial itself.
- $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$
- $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Taylor Series



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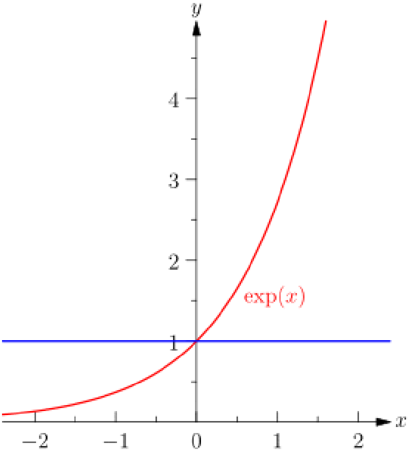
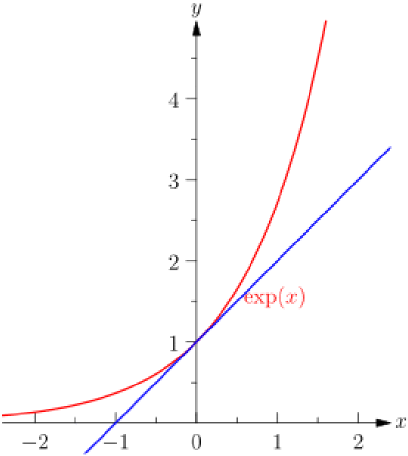


Taylor Series

$1 + x$



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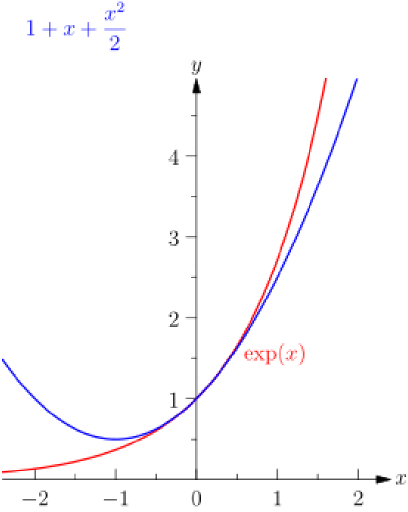

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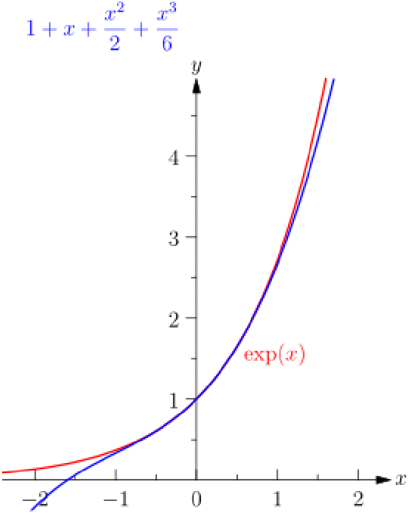

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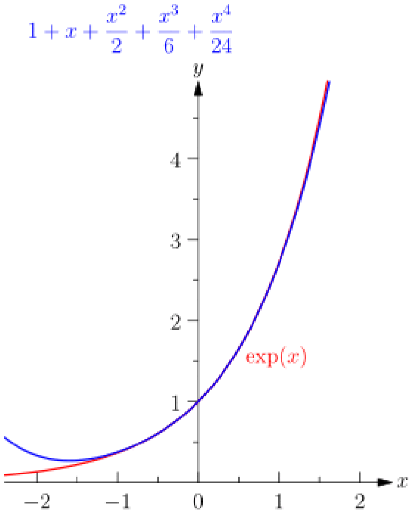

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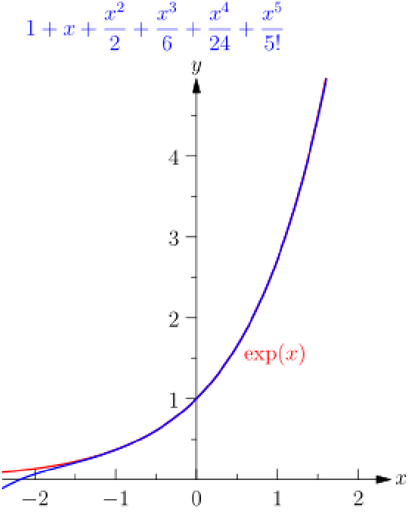

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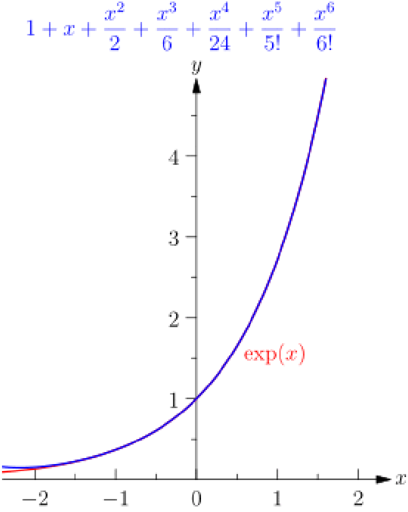

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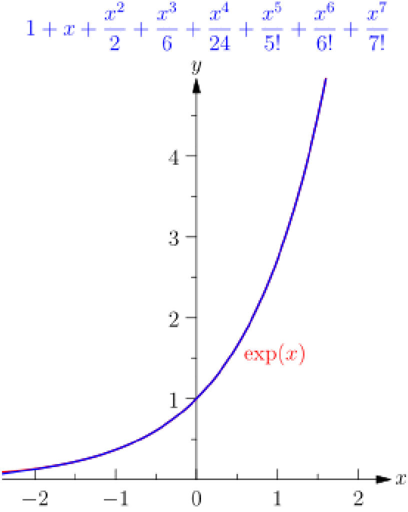

- Try another function yourself!

Indefinite Integral

$2x$ is the derivative of x^2

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So what is x^2?

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The indefinite integral is denoted like this:

$$\int f(x)\,dx$$

Indefinite Integral

So in this notation,

$$\int 2x\,dx = x^2$$

Can you name another function that also has derivative $2x$?

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If $F'(x) = f(x)$, then $(F(x) + c)' = f(x)$ for any number c, so actually:

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$\int f(x)\,dx$ is not one function, but a *set* of functions.

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Properties

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$$\int af(x)\,dx = a\int f(x)\,dx$$

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   This can also be written as:

   $$\int f\,dg = fg - \int g\,df,$$

   where $\int f\,dg = \int fg'\,dx$.

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- The antiderivative of $f(x) = \frac{1}{x}$ is:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

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- The antiderivative of $f(x) = \frac{1}{1+x^2}$ is:

$$\int \frac{1}{1+x^2} \, dx = \operatorname{arctg} x + C$$

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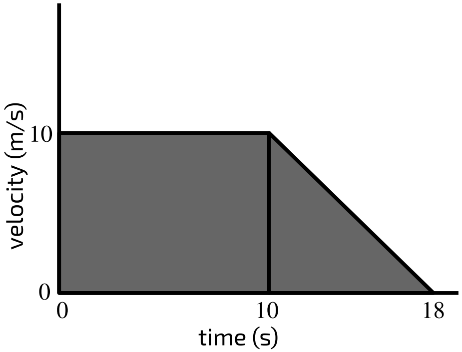
All of this is, of course, computation and tricks (no need to memorize).
How about we make an *actual* use of this?

Definite Integral

Suppose we are given the velocity of a car at each timepoint. How can we calculate the distance travelled by the car?



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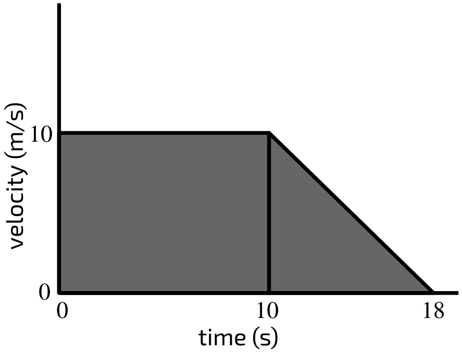

According to physics, distance = area under the velocity curve.

Question

Suppose you have a continuous function f. How can we calculate the area under its graph?

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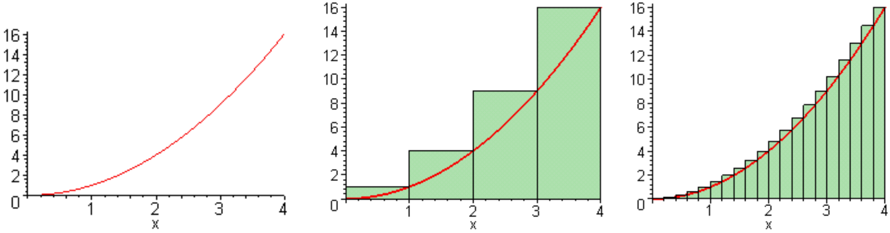
By dividing it into tiny rectangles and adding up their areas.



Definite Integral

Take the interval $[a, b]$ and divide (partition) it into n small parts with points $\{x_0, x_1, \ldots, x_n\}$. Let $\Delta x_i = x_i - x_{i-1}$ denote the length of $[x_{i-1}, x_i]$.

Definite Integral

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The *Riemann sum* of a function $f(x)$ is given by:

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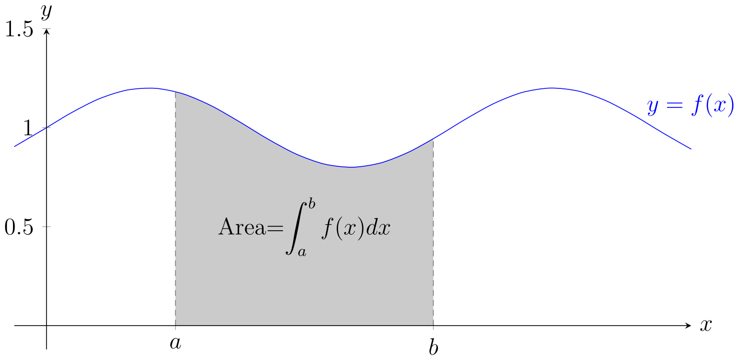
Definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

is called the *definite integral* of the function $f(x)$ on $[a, b]$.

Definite Integral

The definite integral $\int_a^b f(x)\,dx$ represents the **signed area** between the graph of $f(x)$ and the x-axis over the interval $[a, b]$.



- Play with Riemann sums!

Definite Integral

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Theorem (very important)

Suppose that $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is an antiderivative of $f(x)$, then

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Example

- $$\int_0^2 x^2\,dx = \frac{1}{3}\cdot x^3\Big|_0^2 = \frac{1}{3}(2^3 - 0^3) = \frac{8}{3}$$
- $$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = 2$$

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5. $$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

i.e. the name of the variable does not matter.