Eigenvalues, Eigendecomposition, SVD

Hayk Aprikyan, Hayk Tarkhanyan

Motivation

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In general, a diagonal matrix stretches the space in i^{th} direction by d_i times:

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Let's see if we can make any matrix look like this.

Eigenvalues and Eigenvectors

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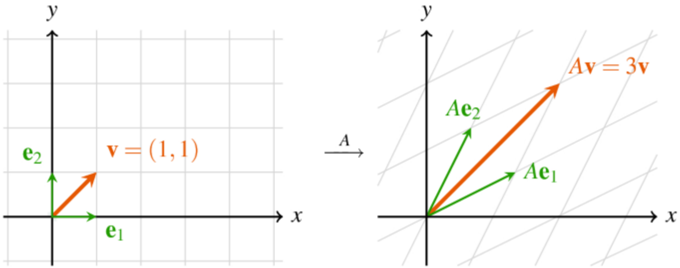
Usually, when we multiply a vector by a square matrix, both its length and direction change.

However, sometimes there are very special vectors which **do not change their direction** when being multiplied/transformed by that matrix:



but only get stretched or squished (or flipped) by some factor.

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you get the same vector $\mathbf{v}$ back, just scaled by some factor λ.

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If for some number λ and some non-zero vector $\mathbf{v}$

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then we say

- λ is an **eigenvalue** of A,
- $\mathbf{v}$ is an **eigenvector** of A corresponding to the eigenvalue λ.

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Note that λ can be any number (including negative numbers and zero).

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For the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4\mathbf{v}$$

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If $\mathbf{v}$ is an eigenvector of A, any multiple of $\mathbf{v}$ is also an eigenvector!

$$A(c \cdot \mathbf{v}) = c \cdot (A\mathbf{v}) = c \cdot (4\mathbf{v}) = 4 \cdot (c\mathbf{v})$$

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Do all matrices have eigenvalues and eigenvectors?

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In fact, for each fixed eigenvalue λ, the set of all eigenvectors corresponding to λ, together with the zero vector $\mathbf{0}$, is a vector subspace of $\mathbb{R}^n$, called the *eigenspace* corresponding to λ:

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The set of all eigenvalues of A is called the *spectrum* of A.

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The matrix $(A - \lambda I)$ squishes down some non-zero vector $\mathbf{v}$ to zero, which means its determinant must be zero:

$$\det(A - \lambda I) = 0$$

Eigenvalues and Eigenvectors

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the expression on the left is a polynomial (where λ is the unknown):

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

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so the determinant is:

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In general, for any $n \times n$ matrix A, the expression $\det(A - \lambda I)$ is a polynomial of degree n in λ.

We call it the *characteristic polynomial* of A and denote it by $p_A(\lambda)$. The roots of this polynomial are the eigenvalues of A.

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The characteristic polynomial is:

$$p_A(\lambda) = \det(A - \lambda I) = (3-\lambda)(-1-\lambda) - 5 = \lambda^2 - 2\lambda - 8$$

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We found the eigenvalues of A! Now, let's find the corresponding eigenvectors.

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from which we get $y = \frac{1}{5}x$, i.e. all the vectors on the line $y = \frac{1}{5}x$ are eigenvectors corresponding to $\lambda = 4$.

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Question

What do you think happens to the eigenvalues of a matrix if we transpose it?

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In this case, we say that eigenvalue $\lambda = 3$ has *algebraic multiplicity* 2, and $\lambda = 5$ has algebraic multiplicity 1.

As a bonus, we have a beautiful and surprising theorem:

Theorem

The determinant of a matrix is equal to the product of its eigenvalues:

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

and the trace of a matrix is equal to the sum of its eigenvalues:

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

Note that the eigenvalues are counted **as many times** as their algebraic multiplicities here.

Eigendecomposition

Suppose now $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix with n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$).

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Since there are n of them, they form a basis of $\mathbb{R}^n$. Therefore any vector $\mathbf{x} \in \mathbb{R}^n$ can be expressed as a linear combination of these eigenvectors:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

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It has the same effect as stretching in each direction of $\mathbf{v}_i$ by λ_i times!

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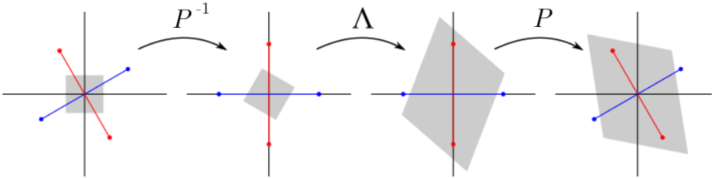
This is called the *eigendecomposition* (or *spectral decomposition*) of A.

Eigendecomposition



Geometrically, the eigendecomposition means that multiplying by A is:

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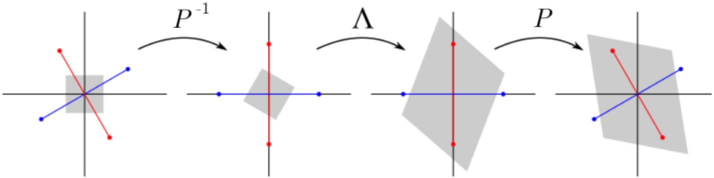

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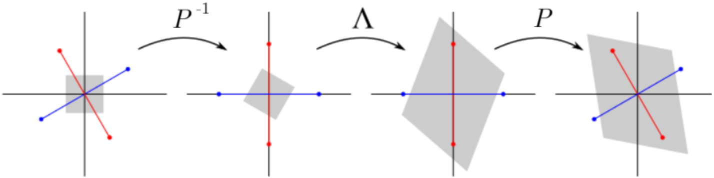

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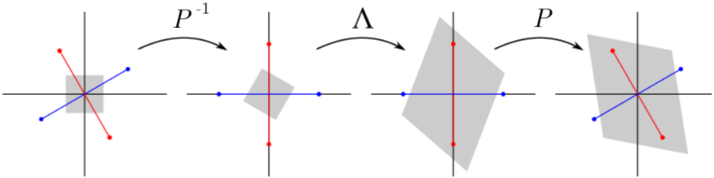

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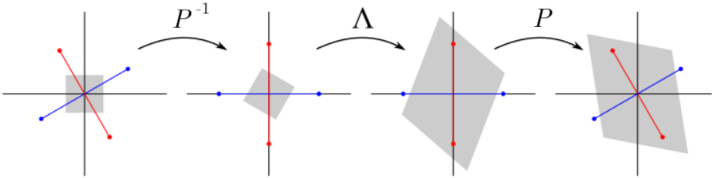

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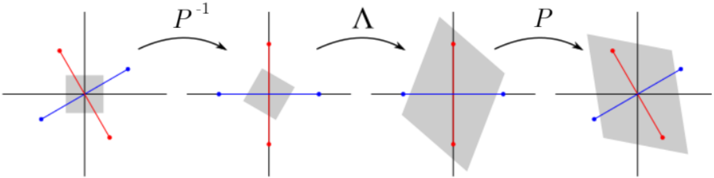

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Question

How would you compute A^{10}?

Singular Value Decomposition (SVD)

The eigendecomposition is very useful, but it only works for square symmetric matrices with enough linearly independent eigenvectors. What if we want to decompose a non-square matrix?

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The answer is the **Singular Value Decomposition (SVD)**:

Theorem

Any matrix $A \in \mathbb{R}^{m \times n}$ can be written as a product of three matrices:

$$A = U \Sigma V^T$$

where

- $U \in \mathbb{R}^{m \times m}$ is a rotation in $\mathbb{R}^m$, and $U^T U = I$
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The numbers in Σ are called the *singular values* of A.

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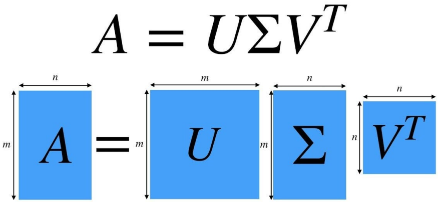

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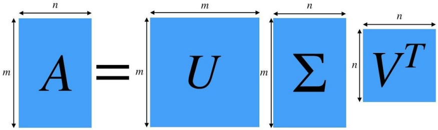

When A is symmetric, the singular values coincide with the eigenvalues of A, and this decomposition (essentially) coincides with the eigendecomposition.

We won't go into the details of how to compute the SVD, but we will see some of its applications later in the course.

Positive and Negative Definiteness

One last thing about eigenvalues. Turns out,

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Similarly, if all eigenvalues are negative (or equivalently, if $\mathbf{x}^T A \mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$), we say that the matrix is **negative definite**:

$$A \prec 0$$

Positive and Negative Definiteness

Example

$A = \begin{bmatrix} 4 & -4 \\ -4 & 5 \end{bmatrix}$ is positive definite because its eigenvalues are

$$\lambda_1 \approx 0.47, \qquad \lambda_2 \approx 8.53$$

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Can we check if a matrix is positive/negative definite without computing its eigenvalues?

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Suppose $A \in \mathbb{R}^{n\times n}$ is a **symmetric** matrix:

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and $A \prec 0$ if and only if they alternate in sign:

$$\det(A_1) < 0, \quad \det(A_2) > 0, \quad \det(A_3) < 0, \quad \ldots$$