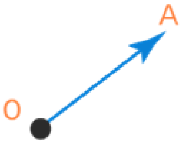
Geometry of Vectors, Matrices

Hayk Aprikyan, Hayk Tarkhanyan

Norm

What if we want to measure the length of some vector?



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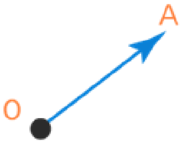

What we can say, is that

the length of the vector

$$=$$

the distance between O and A.

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But how to measure distance?

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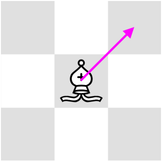
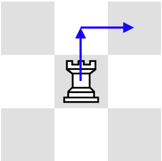


For a bishop, the distance to its upper-right neighbor is 1.



While for a rook, it is 2.

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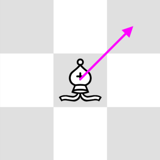
But how to measure distance?



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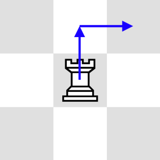

While for a rook, it is 2.

So there are different ways to measure distance and length.

Norm

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, its **Euclidean norm** or **L2 norm** is:

$$\|\mathbf{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

or, equivalently,

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

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Euclidean norm is the standard length we use in classic geometry.
Sometimes we omit the little "2" and just write $\|\mathbf{v}\|$ instead of $\|\mathbf{v}\|_2$.

Norm

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, its **Manhattan norm** or **L1 norm** is:

$$\|\mathbf{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

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Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The Manhattan norm of $\mathbf{v}$ is:

$$\|\mathbf{v}\|_1 = |3| + |4| = 7$$

Norm

As we have seen, there are different types of norms (=many different ways to calculate the length of a vector), and one of them is chosen depending on the problem.

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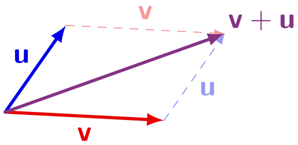
Notice, however, that independently of which one we take, all norms always satisfy the following three properties:

1. $\|\mathbf{v}\| \geq 0$, and equals 0 if only if $\mathbf{v} = \mathbf{0}$,
2. $\|c\mathbf{v}\| = |c| \cdot \|\mathbf{v}\|$,
3. $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$.



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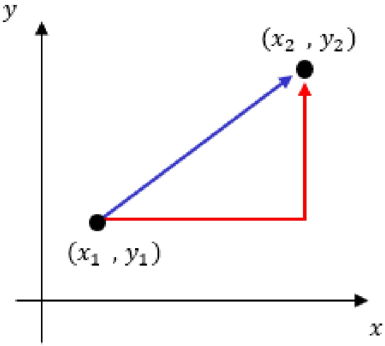

Manhattan Distance L^1

Euclidean Distance L^2

$$L^1 = |x_2 - x_1| + |y_2 - y_1|$$

$$L^2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Angle between vectors

Geometrically, a vector is an arrow in space, that is, it has both a length and direction.

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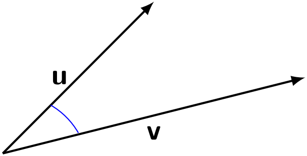

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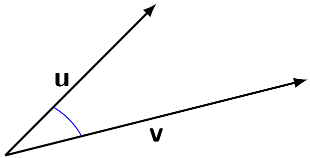

Remember the formula from high school geometry:
$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \alpha$, where α is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Angle between vectors

Definition

The angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is the angle $0 \leq \theta \leq \pi$ for which:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$$

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Find the angle θ between **u** and **v**.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{(3 \cdot 7) + (4 \cdot 1)}{\sqrt{3^2 + 4^2} \cdot \sqrt{7^2 + 1^2}} = \frac{25}{\sqrt{25} \cdot \sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \quad \Rightarrow \quad \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4} = 45^\circ$$

Angle between vectors

Corollary 1

For any vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} \cdot \mathbf{u} = \|\mathbf{v}\| \cdot \|\mathbf{u}\| \cdot \cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$.

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The dot product of two vectors equals 0 if and only if they are perpendicular to each other (form a 90° angle).

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Corollary 3

Any vector $\mathbf{v} \in \mathbb{R}^n$ forms an angle of 0° with itself and 180° with its negative.

Vector Space

Finally, we are left to notice two things. Take, for example,

- the set $D = \{0, 1, 2, \dots, 9\}$ of digits, and
- the set $P = \{x \in \mathbb{R} : x > 0\}$ of positive numbers.

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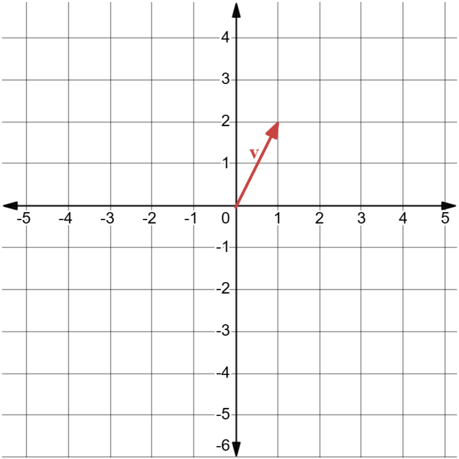
In this case we say that the set of vectors is **closed under addition and scalar multiplication**, while D or P are not (P is closed under addition only).

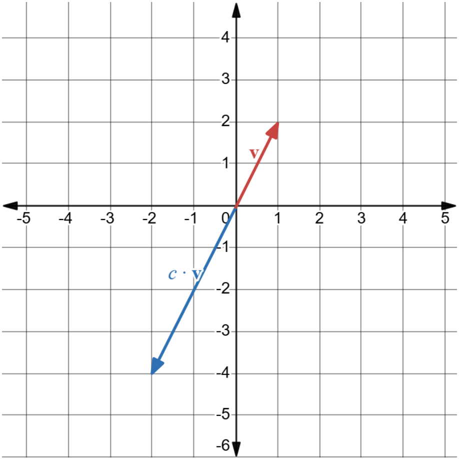
Vector Space

Furthermore, take the line $y = 2x$ and choose any vector on it:



Vector Space

After multiplying it with any number c, it will still stay on the line $y = 2x$:



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In other words, the line $y = 2x$ is **closed under addition and scalar multiplication**, just like the whole set of vectors $\mathbb{R}^2$. This motivates us to give a special name to the good sets like the line $y = 2x$ and $\mathbb{R}^2$.

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We say that $\mathbb{R}^2$ is a **vector space**, and the set of vectors lying on the line $y = 2x$ are a **vector subspace** of $\mathbb{R}^2$.

Vector Space

Definition

A set V is called a **vector space** if

1. it is closed under addition and scalar multiplication,
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
4. There exists a vector $\mathbf{0}$ such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$
5. For every $\mathbf{v}\in V$, there exists a vector $-\mathbf{v}$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$
6. $(cd)\cdot\mathbf{v}=c\cdot(d\cdot\mathbf{v})$
7. $1\cdot\mathbf{v}=\mathbf{v}$
8. $c\cdot(\mathbf{u}+\mathbf{v})=c\cdot\mathbf{u}+c\cdot\mathbf{v}$
9. $(c+d)\cdot\mathbf{v}=c\cdot\mathbf{v}+d\cdot\mathbf{v}$

No need to memorize the properties–just the natural laws of addition and scalar multiplication.

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Theorem

Assume V is a vector space, and U is a subset of V. Then U is a subspace of V if and only if

1. $\mathbf{x} + \mathbf{y} \in U$, for all $\mathbf{x}, \mathbf{y} \in U$,
2. $c\mathbf{x} \in U$, for all $\mathbf{x} \in U$ and $c \in \mathbb{R}$.

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- So $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$ are all vector spaces.
- The set of all vectors that lie on the same line (e.g. $y = kx$) form a subspace (on the condition that the line also contains the $\mathbf{0}$ vector).

Matrices

Definition

An $m \times n$ tuple A of elements a_{ij} ($i = 1, \ldots, m$ and $j = 1, \ldots, n$), is called a real-valued (m, n) **matrix**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R}.$$

The set of all real-valued (m, n) matrices is denoted by $\mathbb{R}^{m \times n}$.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3} \qquad B = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

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Note that the first number in (m, n) always shows rows, second: columns.

Matrix Addition

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Definition

The sum of two matrices A and B, denoted as $A + B$, is obtained by adding corresponding elements. If A is of size $m \times n$ and B is of the same size, then $A + B$ is also of size $m \times n$.

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Remark

Matrix addition is only defined for matrices of the same size.

Scalar Multiplication of a Matrix

Definition

The product of a scalar c and a matrix A, denoted as cA, is obtained by multiplying each element of the matrix by the scalar.

$$c \cdot A = c \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} & \dots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \dots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \dots & c \cdot a_{mn} \end{bmatrix}$$

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Scalar multiplication can be performed for any scalar c and any matrix A.

Negative of a Matrix

Definition

The negative of a matrix A, denoted as $-A$, is obtained by changing the sign of each element in the matrix.

$$-A = -\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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Remark

The negative of a matrix equals (-1) times the matrix.

Matrix Subtraction

Definition

The difference of two matrices A and B, denoted as $A - B$, is obtained by subtracting corresponding elements, or by adding A and $-B$. If A and B are both of size $m \times n$, then $A - B$ is also of size $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

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Example

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Zero Matrix

Definition

The **zero matrix**, denoted as O or $O_{m\times n}$, is a matrix where all elements are zero.

Example

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Remark

$A + O = O + A = A$ for any matrix A.

Transpose of a Matrix

Definition

The **transpose** of a matrix A, denoted as A^T, is obtained by swapping its rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} 7 & 4 & 2 \\ 0 & 1 & -3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 2 & -3 \end{bmatrix}$$

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Remark

The transpose of an (m, n) matrix is an (n, m) matrix.

Matrices

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In that sense, it is not difficult to prove that:

Theorem

For each $m, n \in \mathbb{N}$ the set of real-valued matrices $\mathbb{R}^{m \times n}$ forms a vector space.

Matrix-Vector Multiplication

Definition

Let A be an $m \times n$ matrix and $\mathbf{v}$ be a column vector of size $n \times 1$. The product $A\mathbf{v}$ is a column vector of size $m \times 1$ obtained by multiplying each row of A by the corresponding element of $\mathbf{v}$ and summing the results.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}v_n \end{bmatrix}$$

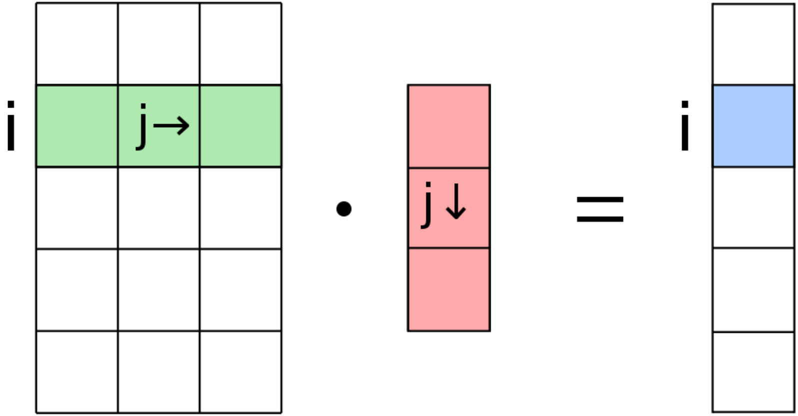
Matrix-Vector Multiplication

Or, in other words, if we denote the rows of A by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$, the product $A\mathbf{v}$ will be a column vector of size $m \times 1$ obtained by taking the dot product of each row of A with the vector $\mathbf{v}$:

$$A = \begin{bmatrix} \ldots & \mathbf{A}_1 & \ldots \\ \ldots & \mathbf{A}_2 & \ldots \\ & \vdots & \\ \ldots & \mathbf{A}_m & \ldots \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}v_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{v} \\ \mathbf{A}_2 \cdot \mathbf{v} \\ \vdots \\ \mathbf{A}_m \cdot \mathbf{v} \end{bmatrix}$$

Matrix-Vector Multiplication



Matrix-Vector Multiplication

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 3 \\ 4 \cdot 2 + 5 \cdot (-1) + 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}$$

Matrix-Vector Multiplication

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Example

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} (-2) \cdot 4 + 1 \cdot 2 \\ 0 \cdot 4 + 3 \cdot 2 \\ 1 \cdot 4 + (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix}$$

Matrix-Vector Multiplication

Matrix-vector multiplication shares properties with scalar multiplication and addition of vectors.

- **Distributive Property:**
  For a matrix A and vectors $\mathbf{v}$ and $\mathbf{w}$ of appropriate sizes:

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$$

- **Scalar Multiplication:**
  For a matrix A and a scalar c:

$$A(c\mathbf{v}) = c(A\mathbf{v})$$

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Note that we can only multiply a matrix by a vector if the number of columns of the matrix equals the length of the vector.

Geometric Interpretation

Why do we define the matrix-vector multiplication this way? Turns out, it has a beautiful geometrical interpretation.

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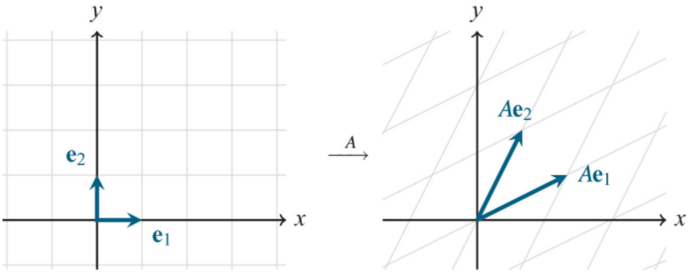
Think this way: when you multiply, say, a 2×2 matrix A by a vector $\mathbf{v} \in \mathbb{R}^2$, what you get is another vector $\mathbf{u} = A\mathbf{v} \in \mathbb{R}^2$. We call this $\mathbf{u}$ the **transformed version** of $\mathbf{v}$ (and we say that A is a linear transformation).



Geometric Interpretation

As we will see later, the resulting "transformed version" **u** is just the same old **v** except it is **rotated** and **scaled** to become longer or shorter (and possibly, flipped).

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As we will see later, the resulting "transformed version" **u** is just the same old **v** except it is **rotated** and **scaled** to become longer or shorter (and possibly, flipped).

In this sense, all matrices are either just rotating vectors by some degree, or flipping them horizontally/vertically, or scale them, or do all three.

The key thing is: whatever a matrix "does" to one vector, it does the same to all other vectors too (when being multiplied with them).

Check different matrices yourself:

- `visualize-it.github.io/linear_transformations/simulation.html`
- `www.shad.io/MatVis`

We will learn more about this later–now back to matrices∼

Matrix Multiplication

Definition

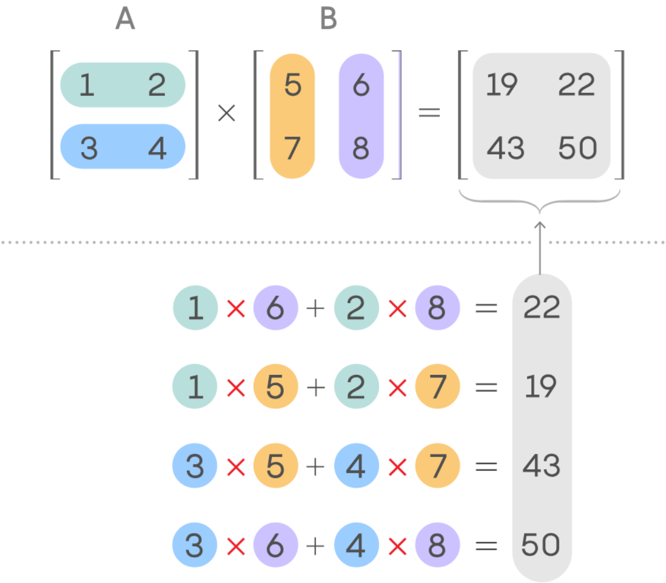
Let A be an $m \times n$ matrix, and let B be an $n \times k$ matrix. The product $C = AB$ is an $m \times k$ matrix, where each element c_{ij} is obtained by taking the dot product of the i-th row of A and the j-th column of B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix}$$

$$\text{where } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{p=1}^{n} a_{ip}b_{pj}$$

Matrix Multiplication



Matrix Multiplication

Matrix multiplication shares properties with scalar multiplication and addition of vectors, as well as matrix-vector multiplication.

- **Distributive Property:**
  For matrices A, B, and C of appropriate sizes:
  $$A(B+C) = AB + AC \quad \text{and} \quad (A+B)C = AC + BC$$
- **Associativity Property:**
  For matrices A, B, and C of appropriate sizes:
  $$A(BC) = (AB)C$$
- **Scalar Multiplication:**
  For matrices A, B of appropriate sizes and a scalar c:
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Note that we can only multiply two matrices if the number of columns of the first matrix equals the number of rows of the second matrix: $(m \times n)$ with $(n \times k)$.

Matrix Multiplication

Example

Let

$$C = \begin{bmatrix} -1 & 0 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2} \qquad D = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & 7 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$CD = \begin{bmatrix} -1\cdot 5 + 0\cdot 3 & -1\cdot(-2) + 0\cdot 0 & -1\cdot 1 + 0\cdot 7 \\ 2\cdot 5 + (-3)\cdot 3 & 2\cdot(-2) + (-3)\cdot 0 & 2\cdot 1 + (-3)\cdot 7 \\ 4\cdot 5 + 1\cdot 3 & 4\cdot(-2) + 1\cdot 0 & 4\cdot 1 + 1\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 & -1 \\ 1 & -4 & -19 \\ 23 & -8 & 11 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$