Angles, Vector Spaces, Matrices

Hayk Aprikyan, Hayk Tarkhanyan

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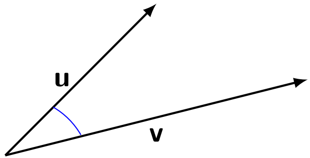

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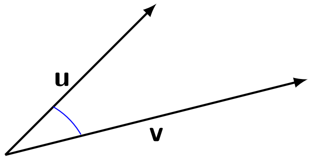

Remember the formula from high school geometry:
$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \alpha$, where α is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Angle between vectors

Definition

The angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is the angle $0 \leq \theta \leq \pi$ for which:

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{(3 \cdot 7) + (4 \cdot 1)}{\sqrt{3^2 + 4^2} \cdot \sqrt{7^2 + 1^2}} = \frac{25}{\sqrt{25} \cdot \sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \quad \Rightarrow \quad \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4} = 45^\circ$$

Angle between vectors

Corollary

For any vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} \cdot \mathbf{u} = \|\mathbf{v}\| \cdot \|\mathbf{u}\| \cdot \cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$.

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i.e. for any numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ we have

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

a fact known as the **Cauchy-Schwarz inequality**.

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This is important because being orthogonal is the mathematical way of saying "pointing in completely different directions".

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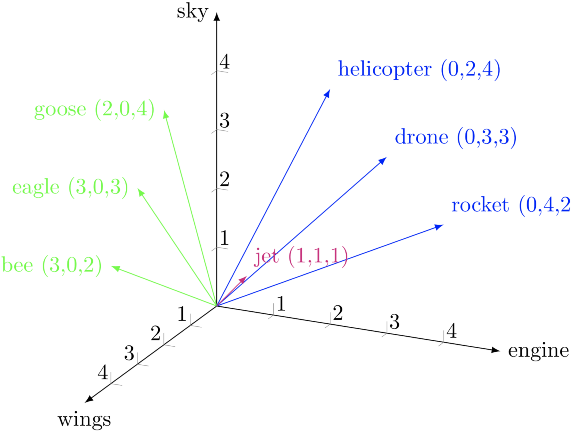

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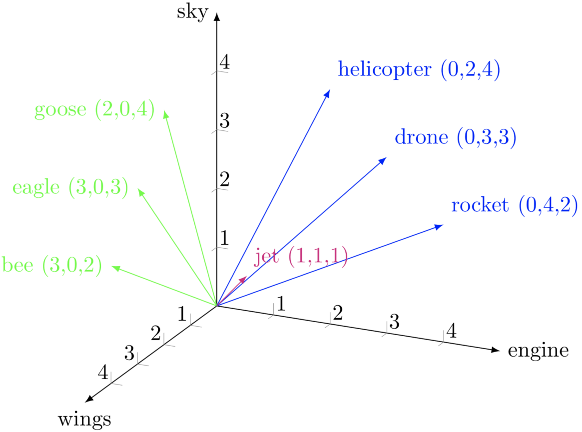

Which vectors are more similar, **drone** and **jet**, or **drone** and **rocket**?

Cosine similarity (optional)

A useful way to measure how similar two vectors are, is to measure the *cosine* of the angle between them:

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While the L1 or L2 distances between two vectors measure how far or how *close* their endpoints are, the cosine similarity measures how *aligned* they are – independently of their lengths.

In the example of **drone**, **jet**, and **rocket**, the cosine between **drone** and **jet** is ≈ 0.95, while the cosine between **drone** and **rocket** is ≈ 0.7.

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(check some actual word embeddings here)

Vector Space

Finally, we are left to notice two things. Take, for example,

- the set $D = \{0, 1, 2, \ldots, 9\}$ of digits, and
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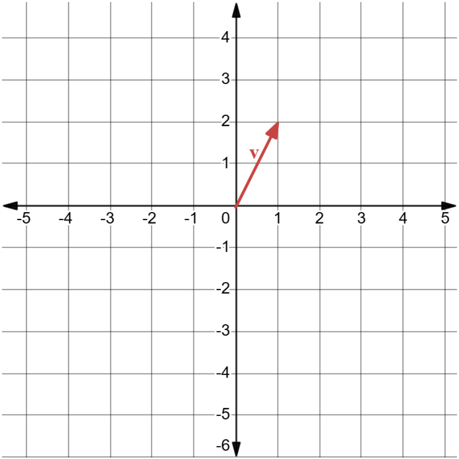
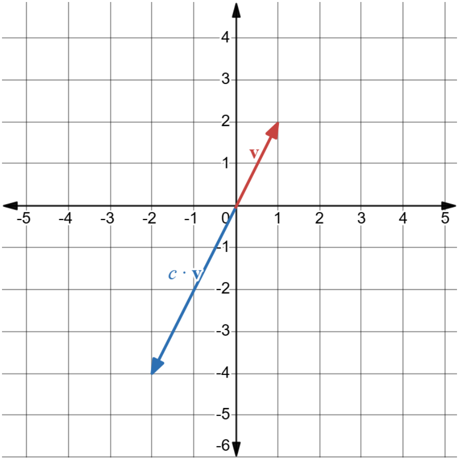
In this case we say that the set of vectors is **closed under addition and scalar multiplication**, while D or P are not (P is closed under addition only).

Vector Space

Furthermore, take the line $y = 2x$ and choose any vector on it:



After multiplying it with any number c, it will still stay on the line $y = 2x$:



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In other words, the line $y = 2x$ is **closed under addition and scalar multiplication**, just like the whole set of vectors $\mathbb{R}^2$. This motivates us to give a special name to the good sets like the line $y = 2x$ and $\mathbb{R}^2$.

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We say that $\mathbb{R}^2$ is a **vector space**, and the set of vectors lying on the line $y = 2x$ are a **vector subspace** of $\mathbb{R}^2$.

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No need to memorize–just the natural laws of addition and multiplication.

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Theorem

Assume V is a vector space, and U is a subset of V. Then U is a subspace of V if and only if

1. $\mathbf{x} + \mathbf{y} \in U$, for all $\mathbf{x}, \mathbf{y} \in U$,
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- So $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, ... are all vector spaces.
- The set of all vectors that lie on the same line (e.g. $y = kx$) form a subspace (on the condition that the line also contains the $\mathbf{0}$ vector).

Matrices

Definition

An $m \times n$ table A of elements a_{ij} is called an (m, n) **matrix**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R}$$

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Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3} \qquad B = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

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Note that the first number in (m, n) **always** shows rows, second – columns.

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Definition

The sum of two matrices A and B, denoted as $A+B$, is obtained by adding corresponding elements. If A is of size $m\times n$ and B is of the same size, then $A+B$ is also of size $m\times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

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Question

What do you think happens if we multiply a matrix by a scalar?

Scalar Multiplication of a Matrix

Definition

If A is a matrix and c is any scalar, we define the product cA as the matrix:

$$c \cdot A = c \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} & \dots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \dots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \dots & c \cdot a_{mn} \end{bmatrix}$$

i.e. each element of the matrix is multiplied by the scalar c.

Scalar Multiplication of a Matrix

Definition

If A is a matrix and c is any scalar, we define the product cA as the matrix:

$$c \cdot A = c \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} & \dots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \dots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \dots & c \cdot a_{mn} \end{bmatrix}$$

i.e. each element of the matrix is multiplied by the scalar c.

Scalar multiplication can be performed for any scalar c and any matrix A.

Negative of a Matrix

Similarly, we define the negative of a matrix to be the matrix with all elements multiplied by -1:

Definition

The negative of a matrix A, denoted as $-A$, is obtained by changing the sign of each element in the matrix.

$$-A = -\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} -a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & -a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{m1} & -a_{m2} & \dots & -a_{mn} \end{bmatrix}$$

Matrix Subtraction

Definition

The difference of two matrices A and B (both of the same size), denoted as $A - B$, is obtained by subtracting corresponding elements:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots & a_{2n} - b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \dots & a_{mn} - b_{mn} \end{bmatrix}$$

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What happens if we add a matrix to its negative?

Zero Matrix

Definition

The **zero matrix**, denoted as O or $O_{m\times n}$, is a matrix where all elements are zero.

Example

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Of course, adding O has no effect on a matrix:

Remark

For any zero matrix O of the same size as matrix A, we have

$$A + O = O + A = A$$

Transpose of a Matrix

Definition

The **transpose** of a matrix A, denoted as A^T, is obtained by swapping its rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} 7 & 4 & 2 \\ 0 & 1 & -3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 2 & -3 \end{bmatrix}$$

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If A is an (m, n) matrix, then A^T is an (n, m) matrix.

Matrices

Matrices can be added together and multiplied by numbers, and these operations share the same "good" properties (e.g. $A + B = B + A$) with vectors.

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In that sense, it is not difficult to prove that:

Theorem

For each fixed (m, n), the set $\mathbb{R}^{m \times n}$ of all $m \times n$ matrices forms a vector space.

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Theorem

For each fixed (m, n), the set $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices forms a vector space.

Okay, but why do we need matrices at all?

Matrix-Vector Multiplication

Let's define the multiplication of a matrix by a vector.

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Say, we have a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{v} \in \mathbb{R}^n$:

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As a result we get a new vector of size $m \times 1$.

Matrix-Vector Multiplication

Definition

Let A be an $m \times n$ matrix and $\mathbf{v}$ be a column vector of size $n \times 1$. The product $A\mathbf{v}$ is a column vector of size $m \times 1$ obtained by multiplying each row of A by the corresponding element of $\mathbf{v}$ and summing the results.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

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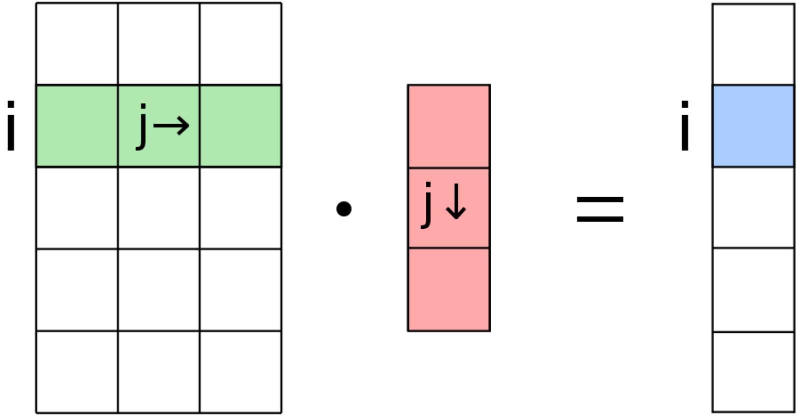
Matrix-Vector Multiplication



Matrix-Vector Multiplication

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 3 \\ 4 \cdot 2 + 5 \cdot (-1) + 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} (-2) \cdot 4 + 1 \cdot 2 \\ 0 \cdot 4 + 3 \cdot 2 \\ 1 \cdot 4 + (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix}$$

Matrix-Vector Multiplication

Matrix-vector multiplication shares properties with scalar multiplication and addition of vectors.

- **Distributive Property:**
  For a matrix A and vectors $\mathbf{v}$ and $\mathbf{w}$ of appropriate sizes:

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$$

- **Scalar Multiplication:**
  For a matrix A and a scalar c:

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Note that we can only multiply a matrix by a vector if the number of columns of the matrix equals the length of the vector
(i.e. **not all** matrices and vectors can be multiplied)!

Geometric Interpretation

Why do we define the matrix-vector multiplication this way? Turns out, it has a beautiful geometrical interpretation.

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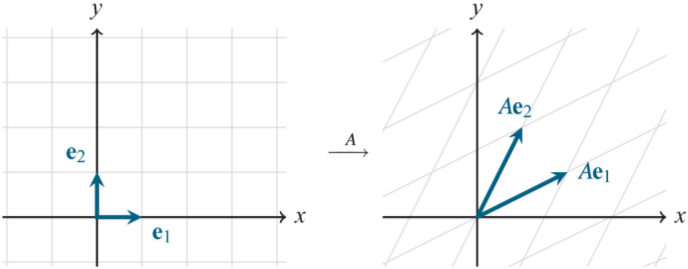
Think this way: when you multiply, say, a 2×2 matrix A by a vector $\mathbf{v} \in \mathbb{R}^2$, what you get is another vector $\mathbf{u} = A\mathbf{v} \in \mathbb{R}^2$. We call this $\mathbf{u}$ the **transformed version** of $\mathbf{v}$ (and we say that A is a linear transformation).



Geometric Interpretation

As we will see later, the resulting "transformed version" $\mathbf{u}$ is just the same old $\mathbf{v}$ except it is **rotated** and **scaled** to become longer or shorter (and possibly, flipped).

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As we will see later, the resulting "transformed version" **u** is just the same old **v** except it is **rotated** and **scaled** to become longer or shorter (and possibly, flipped).

In this sense, all matrices are either just rotating vectors by some degree, or flipping them horizontally/vertically, or scale them, or do all three.

The key thing is: whatever a matrix "does" to one vector, it does the same to all other vectors too (when being multiplied with them).

Check different matrices yourself:
- `visualize-it.github.io/linear_transformations/simulation.html`
- `www.shad.io/MatVis`

We will learn more about this later–now back to matrices∼

Matrix Multiplication

Definition

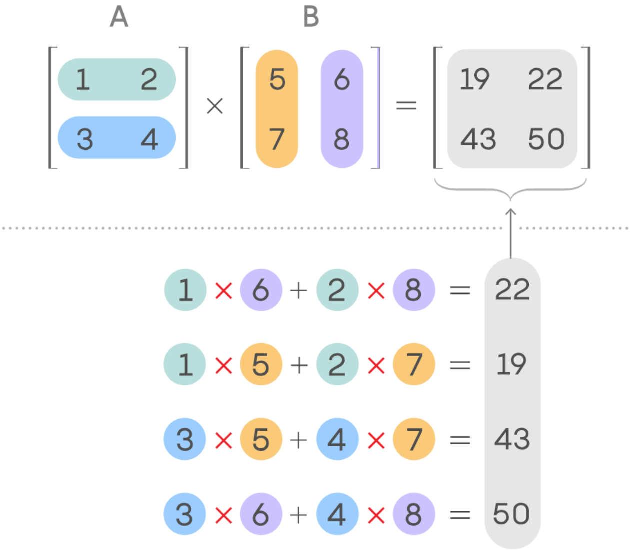
Let A be an $m \times n$ matrix, and let B be an $n \times k$ matrix. The product $C = AB$ is an $m \times k$ matrix, where each element c_{ij} is the dot product of the i^{th} row of A and the j^{th} column of B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.

Matrix Multiplication



Matrix Multiplication

Matrix multiplication shares properties with scalar multiplication and addition of vectors, as well as matrix-vector multiplication.

- **Distributive Property:**

$$A(B+C) = AB + AC \quad \text{and} \quad (A+B)C = AC + BC$$

- **Associative Property:**

$$A(BC) = (AB)C$$

- **Scalar Multiplication:**

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$$A(cB) = c(AB) = (cA)B$$

Note that we can only multiply A and B if the (number of columns in A) equals the (number of rows in B): $(m \times n)$ with $(n \times k)$.

In particular, this means that matrix multiplication is **not commutative**, i.e. in general, $AB \neq BA$!

Matrix Multiplication

Example

Let

$$C = \begin{bmatrix} -1 & 0 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2} \qquad D = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & 7 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$CD = \begin{bmatrix} -1\cdot 5 + 0\cdot 3 & -1\cdot(-2) + 0\cdot 0 & -1\cdot 1 + 0\cdot 7 \\ 2\cdot 5 + (-3)\cdot 3 & 2\cdot(-2) + (-3)\cdot 0 & 2\cdot 1 + (-3)\cdot 7 \\ 4\cdot 5 + 1\cdot 3 & 4\cdot(-2) + 1\cdot 0 & 4\cdot 1 + 1\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 & -1 \\ 1 & -4 & -19 \\ 23 & -8 & 11 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Matrix Multiplication

Example

Let

$$C = \begin{bmatrix} -1 & 0 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2} \qquad D = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & 7 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$CD = \begin{bmatrix} -1\cdot 5 + 0\cdot 3 & -1\cdot(-2) + 0\cdot 0 & -1\cdot 1 + 0\cdot 7 \\ 2\cdot 5 + (-3)\cdot 3 & 2\cdot(-2) + (-3)\cdot 0 & 2\cdot 1 + (-3)\cdot 7 \\ 4\cdot 5 + 1\cdot 3 & 4\cdot(-2) + 1\cdot 0 & 4\cdot 1 + 1\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 & -1 \\ 1 & -4 & -19 \\ 23 & -8 & 11 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

What about DC?