Vectors

Hayk Aprikyan, Hayk Tarkhanyan

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Coins	**Quantity**
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20	1
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200	1

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- How can we denote that?
- Using a table:

Coins	**Quantity**
10	2
20	1
50	2
100	0
200	1

- Or by taking the two columns of the table: $\begin{bmatrix} 10 \\ 20 \\ 50 \\ 100 \\ 200 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Vectors

Definition

An ordered set of n real numbers is called a **vector** (or **column vector**) in $\mathbb{R}^n$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where $v_1, v_2, \ldots, v_n$ are the **components** of the vector.
A vector written horizontally is called a **row vector**:

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

We will denote $\mathbf{v} \in \mathbb{R}^n$ to indicate that $\mathbf{v}$ is a vector in $\mathbb{R}^n$.

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Vectors in $\mathbb{R}^1$ are real numbers: $[v] \in \mathbb{R}$.

Examples of Vectors in $\mathbb{R}^n$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{(3-dimensional column vector)}$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(4-dimensional column vector)}$$

$$\mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \text{(2-dimensional column vector)}$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(Zero vector in 3-dimensional space)}$$

$$\mathbf{v}_5 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \quad \text{(3-dimensional row vector)}$$

Addition of vectors

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3×100 drams and 1×200 drams?

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We would have the following coins:

$$\mathbf{b} + \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 1+0 \\ 2+0 \\ 0+3 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

Addition of vectors

Definition

To add two vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ in $\mathbb{R}^n$, add their corresponding components:

$$\mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

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Note that we can only add two vectors if they are of the same length!

Multiplication of vector by scalar

What if the money in our pockets doubled?

Multiplication of vector by scalar

What if the money in our pockets doubled? We would have:

$$2 \cdot \mathbf{b} = 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

from each coin.

Multiplication of vector by scalar

Definition

To multiply a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ by a scalar c in $\mathbb{R}^n$, multiply each component of the vector by the scalar:

$$c \cdot \mathbf{v} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_n \end{bmatrix}$$

Properties of Vectors

Associativity and Commutativity of Vector Addition

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the vector addition is commutative and associative:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

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Associativity and Commutativity of Scalar Multiplication

For any scalar c and vectors $\mathbf{v}$ and $\mathbf{u}$ in $\mathbb{R}^n$, scalar multiplication is associative and commutative:

$$c \cdot (\mathbf{v} + \mathbf{u}) = c \cdot \mathbf{v} + c \cdot \mathbf{u}$$

$$(c \cdot d) \cdot \mathbf{v} = c \cdot (d \cdot \mathbf{v})$$

Vector Subtraction

What if we buy something and spend 2 x 50 drams?

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Definition

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, the **negative** of $\mathbf{v}$, denoted as $-\mathbf{v}$, is obtained by negating each component:

$$-\mathbf{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$$

Vector Subtraction

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The subtraction of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined as the sum of $\mathbf{u}$ and the negative of $\mathbf{v}$:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{bmatrix}$$

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Example

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 - 1 \\ -1 - 4 \\ 3 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$$

Vector Transposition

Definition

For a column vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, the **transpose**, denoted as $\mathbf{v}^T$, is a row vector:

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

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For a row vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ in $\mathbb{R}^n$, the **transpose**, denoted as $\mathbf{u}^T$, is a column vector:

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Transpose Properties

- For any vector $\mathbf{v}$ in $\mathbb{R}^n$, $(\mathbf{v}^T)^T = \mathbf{v}$
- For any scalar c, $(c \cdot \mathbf{v})^T = c \cdot \mathbf{v}^T$

Dot Product of Vectors

In our example we had $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ coins of $\mathbf{a} = \begin{bmatrix} 10 \\ 20 \\ 50 \\ 100 \\ 200 \end{bmatrix}$ nominations (values) respectively.

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Definition

The **dot product** of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is:

$$\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

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If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$, then:

$$\mathbf{u} \cdot \mathbf{v} = (2 \cdot 1) + (-1 \cdot 4) + (3 \cdot 0) = 2 - 4 + 0 = -2$$

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Going back to our example, we can calculate our money with the dot product of **a** and **b**:

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 20 \\ 50 \\ 100 \\ 200 \end{bmatrix} = 2 \cdot 10 + 1 \cdot 20 + 2 \cdot 50 + 0 \cdot 100 + 1 \cdot 200 = 340$$

Dot Product of Vectors

Remark 1

The dot product of two vectors is defined if and only if the vectors have the same number of components (i.e. are of the same length).

Remark 2

The dot product of two vectors is a *number* (scalar), not a vector.

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Remark 2

The dot product of two vectors is a *number* (scalar), not a vector.

This is why the dot product is often called **scalar product**.

Properties of Dot Product

Properties

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. The dot product has the following properties:

1. Commutativity:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

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4. Non-negativity:
$$\mathbf{u} \cdot \mathbf{u} \geq 0 \text{ and } \mathbf{u} \cdot \mathbf{u} = 0 \text{ if and only if } \mathbf{u} = \mathbf{0}$$

Examples

Consider vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

Let's calculate $(5\mathbf{u} - \mathbf{v}) \cdot \mathbf{w}$:

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Let's calculate $(5\mathbf{u} - \mathbf{v}) \cdot \mathbf{w}$:

$$\begin{aligned} (5\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} &= \left(5 \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\ &= \left(\begin{bmatrix} 5 \\ -10 \\ 15 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -14 \\ 16 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = 5 \cdot (-2) + (-14) \cdot 1 + 16 \cdot 2 = 8 \end{aligned}$$

Geometric interpretation of vectors

So far, we were treating vectors in $\mathbb{R}^n$ as lists of numbers only. Take, for example, the 2d vector

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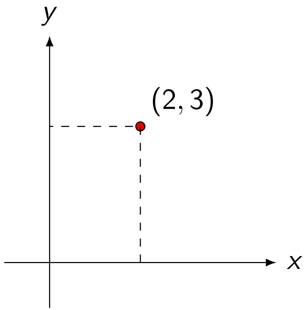

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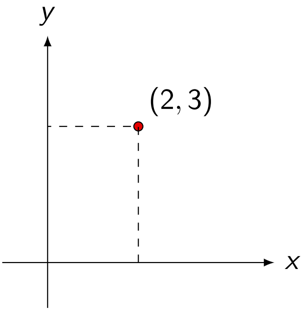

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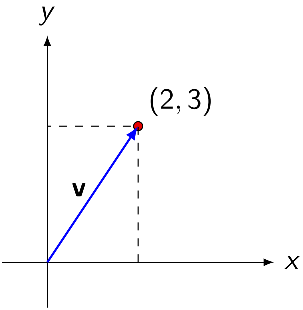

- or as an arrow in space, pointing from $(0, 0)$ to the mentioned point.

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In other words, every 2d vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is essentially an **arrow** starting from the origin $(0, 0)$ and pointing to the point (x, y),

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Question

What do you think happens in the 3d space?

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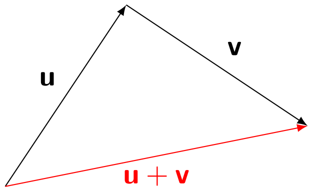
Question

What do you think happens in the 3d space? What about higher dimensions?

Addition of vectors

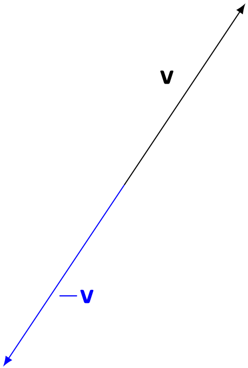
Let's interpret some of our vector operations geometrically.

- **Addition:** To add vectors $\mathbf{u}$ and $\mathbf{v}$, place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector pointing from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$.



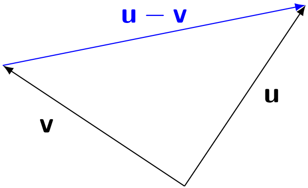
Negative of vectors

- **Negation:** The negative of a vector $\mathbf{v}$, denoted $-\mathbf{v}$, is a vector with the same magnitude but opposite direction.



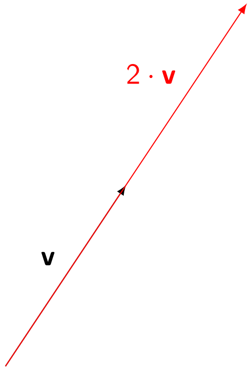
Subtraction of vectors

- **Subtraction:** To subtract **v** from **u**, place them at the same point. Then connect the tail of **v** to the tail of **u**.



Multiplication by scalar

- **Scalar Multiplication:** Scaling a vector $\mathbf{v}$ by a scalar c stretches or compresses the vector. The result $c \cdot \mathbf{v}$ has the same direction as $\mathbf{v}$ but a different magnitude.



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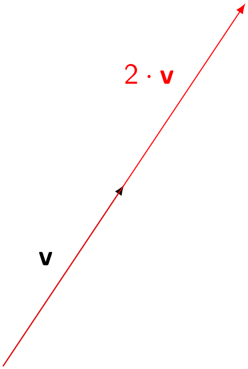

What do you think happens if c is negative?

Example

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Let $\mathbf{a} = [3, 2]$ and $\mathbf{b} = [2, 0]$. We want to find $3\mathbf{a} + \mathbf{b}$.

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Let $\mathbf{a} = [3, 2]$ and $\mathbf{b} = [2, 0]$. We want to find $3\mathbf{a} + \mathbf{b}$.

Algebraically:

$$
\begin{aligned}
3\mathbf{a} + \mathbf{b} &= 3 \cdot [3, 2] + [2, 0] \\
&= [9, 6] + [2, 0] \\
&= [11, 6]
\end{aligned}
$$

Example

Example

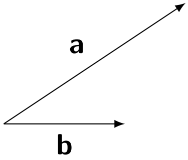
Let $\mathbf{a} = [3, 2]$ and $\mathbf{b} = [2, 0]$. We want to find $3\mathbf{a} + \mathbf{b}$.

Algebraically:

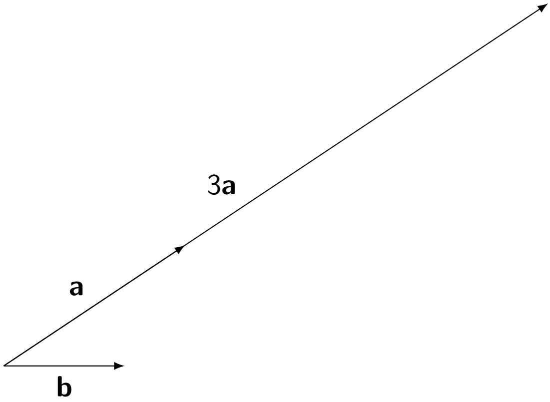
$$\begin{aligned} 3\mathbf{a} + \mathbf{b} &= 3 \cdot [3, 2] + [2, 0] \\ &= [9, 6] + [2, 0] \\ &= [11, 6] \end{aligned}$$

How can we interpret it geometrically?

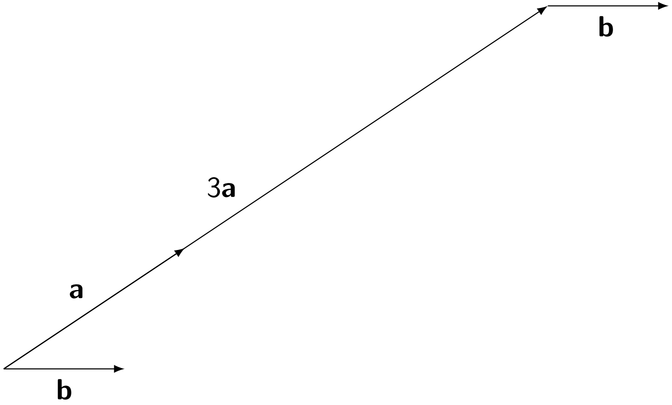
Example



Example



Example



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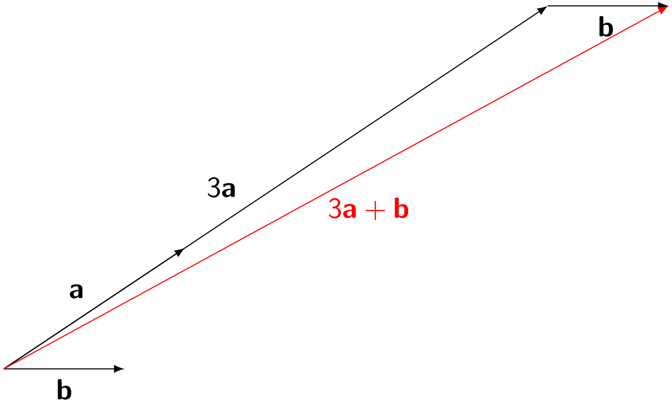
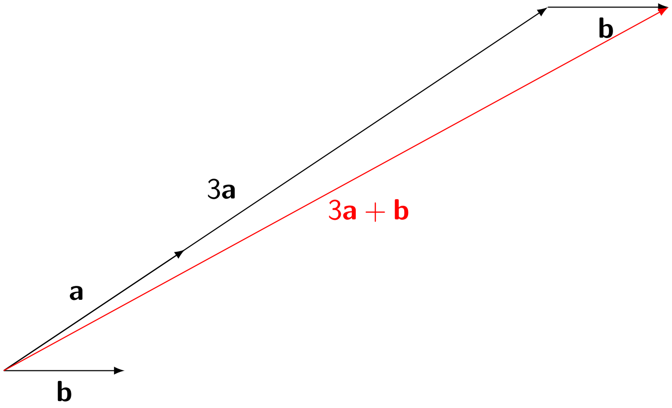


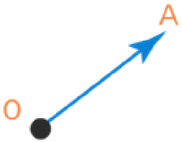
Example



It is like asking directions and being instructed to go 3 steps in the direction of **a**, and then 1 step in the direction of **b**.

Norm

What if we want to measure the length of some vector?



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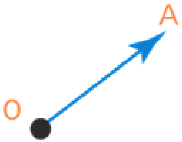

What we can say, is that

the length of the vector

=

the distance between O and A.

Norm

But how to measure distance?

Norm

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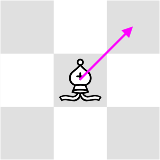
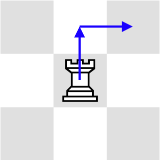


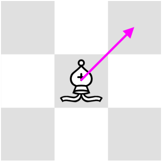
For a bishop, the distance to its upper-right neighbor is 1.



While for a rook, it is 2.

Norm

But how to measure distance?



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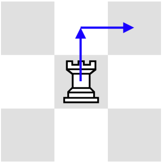

While for a rook, it is 2.

So there are **different ways** to measure distance and length.

Norm

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, its **Euclidean norm** or **L2 norm** is:

$$\|\mathbf{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

or, equivalently,

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

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Example

Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The Euclidean norm of $\mathbf{v}$ is:

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Euclidean norm is the standard length we use in classic geometry. Sometimes we omit the little "2" and just write $\|\mathbf{v}\|$ instead of $\|\mathbf{v}\|_2$.

Norm

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, its **Manhattan norm** or **L1 norm** is:

$$\|\mathbf{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

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Example

Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The Manhattan norm of $\mathbf{v}$ is:

$$\|\mathbf{v}\|_1 = |3| + |4| = 7$$

Norm

As we have seen, there are different types of norms (=many different ways to calculate the length of a vector), and one of them is chosen depending on the problem.

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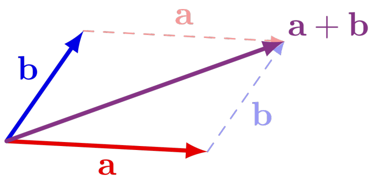
As we have seen, there are different types of norms (=many different ways to calculate the length of a vector), and one of them is chosen depending on the problem.

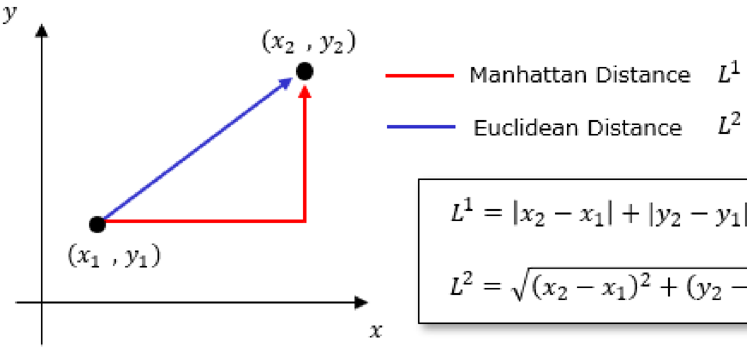
Notice, however, that independently of which one we take, all norms always satisfy the following three properties:

1. $\|\mathbf{v}\| \geq 0$, and equals 0 if only if $\mathbf{v} = \mathbf{0}$,
2. $\|c\mathbf{v}\| = |c| \cdot \|\mathbf{v}\|$,
3. $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$.



Norm

Using the concept of norm, we can now measure the **distance** between two points as the length of the vector connecting them:



$$L^1 = |x_2 - x_1| + |y_2 - y_1|$$

$$L^2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$