Sets, Functions, Combinatorics

Hayk Aprikyan, Hayk Tarkhanyan

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   Important: Ask questions!

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The main resources can be found here:

- Textbooks
- Very short manual (in Armenian)
- More materials for further reading (see here)

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Usually we denote sets by capital letters, Suppose we have a collection of objects. How can we write them down?

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Usually we denote sets by capital letters, so:

 $\in A, \qquad 4 \in C$

but  $\notin A$, $10 \notin C$.

Sets

If all elements of A also belong to B, we say that A is a **subset** of B (and B is a **superset** of A), and write $A \subset B$:

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Note that in sets, the order of elements does not matter:

$$\{1, 4, 7\} = \{7, 1, 4\}$$

and we ignore repeated elements:

$$\{1, 4, 4, 4, 7, 7\} = \{1, 4, 7\}$$

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Some special sets that we will often use are:

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- **integers:** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
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Operations with Sets

Suppose we have two sets A and B:

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We denote the elements that belong to

- *both* A and B by $A \cap B$:

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If $A \cap B = \varnothing$, we say that A and B are **disjoint** (i.e. they have no common elements).

Operations with Sets

Example

If

$$A = \{1, 2, 3, 4, 5\}$$

and

$$B = \{4, 5, 6, 7, 8\}$$

find $A \cap B$ and $A \cup B$.

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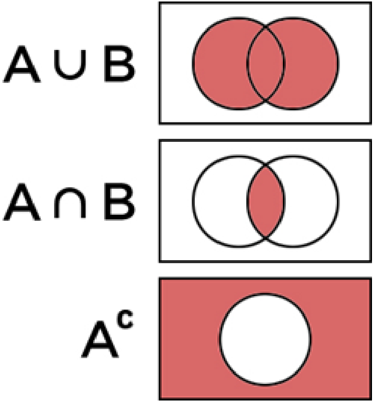
$$A^c = (-\infty, 0] \cup [7, +\infty)$$

We denote the set of those elements by A^c and call the **complement** of A.

Operations with Sets

Graphically, it is sometimes convenient to represent sets by diagrams, called **Venn diagrams**:



Union $A \cup B$:
All elements that belong to A or B or both

Intersection $A \cap B$:
All elements that belong to *both* A and B

Complement A^c:
All elements that do not belong to A

Getting back to our example sets:

$A = \{$ ,  $\}$, $B = \{$ , , ,  $\}$

How much is $|A \cup B|$?

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Indeed,

$A \cup B = \{$🥤, 🏐, 🚙, 🧀, 🛋$\}$

has 5 elements.

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This elegant law is called the **inclusion-exclusion principle**:

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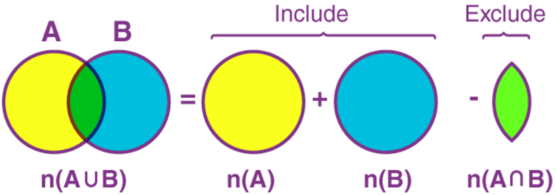

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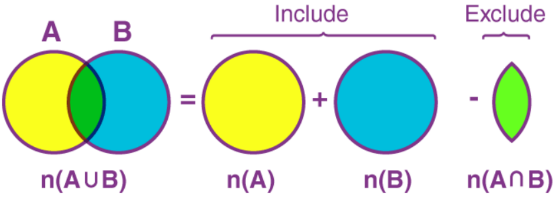

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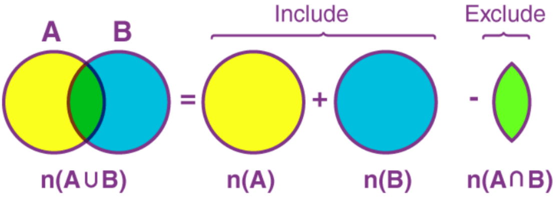

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What if there are three sets A, B and C?

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Assume $A = \{$, $\}$. How many pairs (a_1, a_2) can we form such that $a_1, a_2 \in A$?

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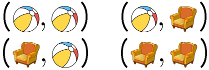

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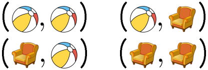

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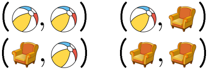

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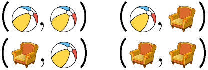

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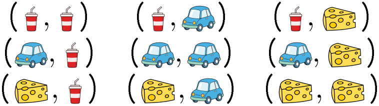

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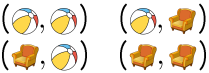

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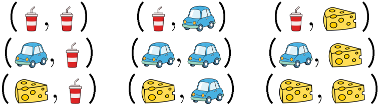

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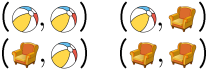

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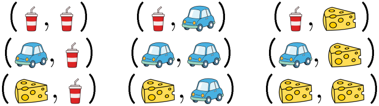

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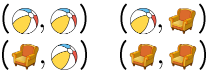

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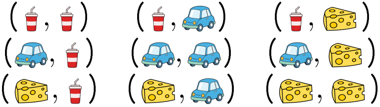

In total, 9 such pairs. Do you see a pattern?

Question

How many two-letter words can you form using the letters in *"movie"*?

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If there are n elements in a set A, the number of pairs (a_1, a_2) *such that* $a_1, a_2 \in A$ *is* n^2.

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Example

Suppose

- $V = \{2, 3, 4, \ldots, 10, J, Q, K, A\}$
- $S = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$

What is $V \times S$?

Cartesian Product

Given two sets A and B, the set of all pairs (a, b) (where $a \in A$ and $b \in B$) is called their **Cartesian product**:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Note that the order of elements in the pair matters, i.e. (, 1) ≠ (1, ).

Example

Suppose

- $V = \{2, 3, 4, \ldots, 10, J, Q, K, A\}$
- $S = \{♥, ♦, ♣, ♠\}$

What is $V \times S$? How many elements does it have?

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Theorem

If $|A| = n$ and $|B| = m$, then $|A \times B| = n \cdot m$.

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If A and B are intervals, e.g. $A = [1, 5]$ and $B = [2, 5]$, then

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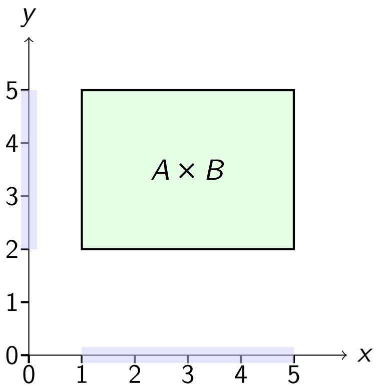
i.e. their Cartesian product is a rectangle:



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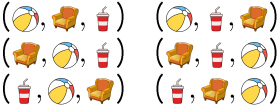

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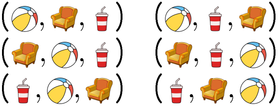

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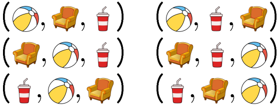

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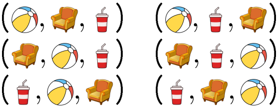

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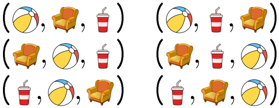

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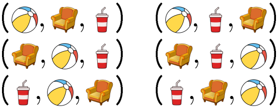

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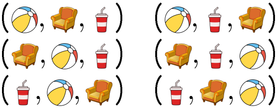

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which can also be written as

$$\frac{n!}{(n-k)!}$$

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What if the order does not matter, i.e. we only want to choose a team of 3 people out of 10 employees (everyone with the same position)?

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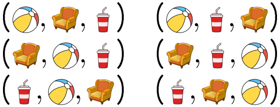

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Suppose A = {🏐, 🛋️, 🥤, 🚙, 🧀}. How many ways can we choose 3 elements from A if the order does not matter?

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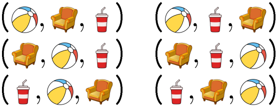

Since in 60 we have counted each group of 3 elements 6 times, we have to divide 60 by 6 to get rid of them:

$$\frac{60}{3 \times 2 \times 1} = 10 \qquad \text{or} \qquad \frac{5!}{2! \cdot 3!} = 10$$

In general,

Theorem

The number of ways to choose k elements from n elements (without any particular order) is

$$\frac{n!}{(n-k)! \cdot k!}$$

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To sum up, the number of ways to choose k elements from n elements is:

- $\frac{n!}{(n-k)!}$ if the order matters
- $\frac{n!}{(n-k)! \cdot k!}$ if the order does not matter

Functions

Finally, let's talk about functions.

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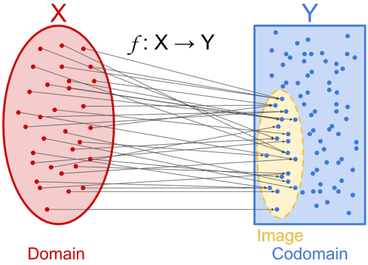

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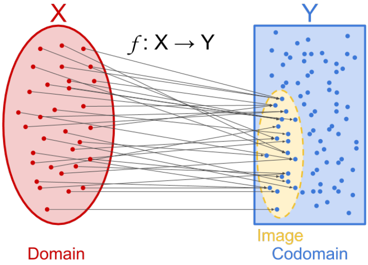

If we assign to each element $x \in X$ **exactly** one element $y \in Y$, we say that we have defined a **function** from X to Y:

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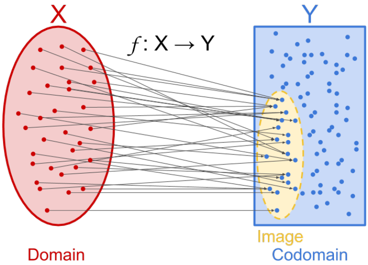

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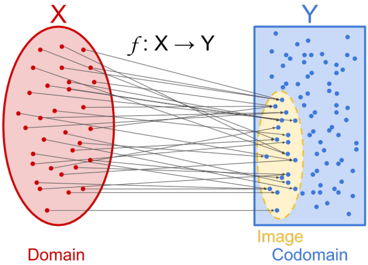

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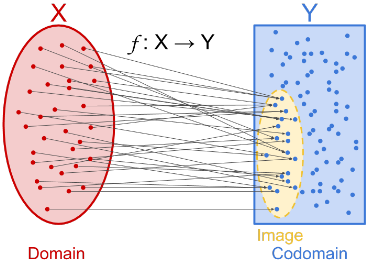

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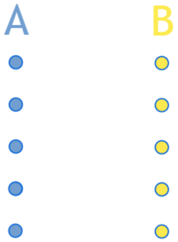
The value $f(x)$ which corresponds to x is called the *image* of x, and x is called the *preimage* of $f(x)$.

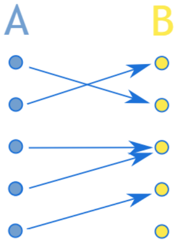
Functions

For example, assume A is the set of all students in one class, and B is the set of another class.



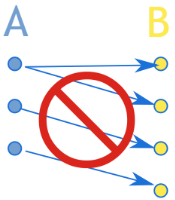
Functions

If each student in class A has a crush on exactly one student in class B, then we have defined a function from A to B:



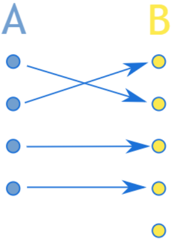
Functions

No student in class A can have a crush on two different students in class B (otherwise it would not be a function):



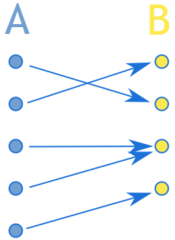
Functions

If no two students in class A have a crush on the same student in class B, then we say that the function is **injective** (or one-to-one):



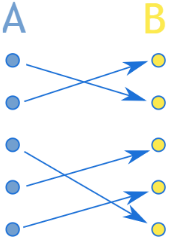
Functions

If to every student in class B there is at least one student in class A who has a crush on them, then we say that the function is **surjective** (or onto):



Functions

If f is both injective and surjective, then we say that it is **bijective**:



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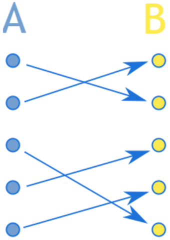

In this case, we can make pairs (couples) of students from A and B, and we can also define the inverse function

$$f^{-1} : B \to A$$

assigning to each student in B the one in A who has a crush on them.

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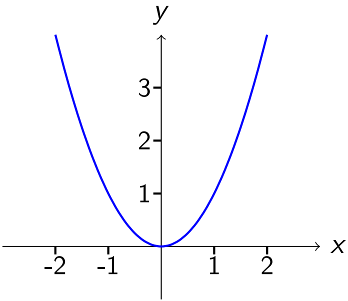

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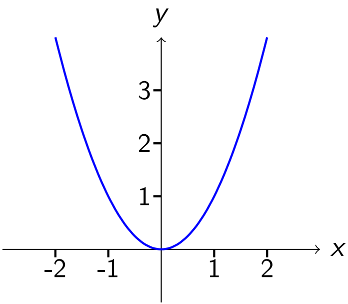

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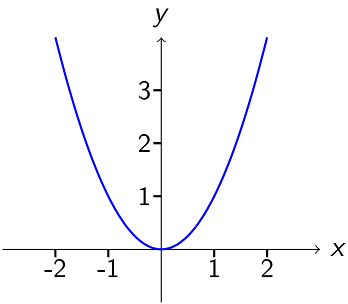

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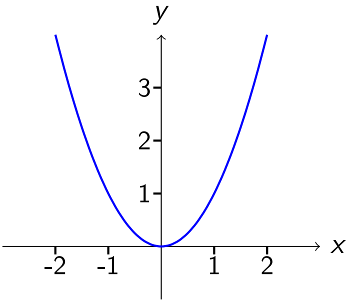

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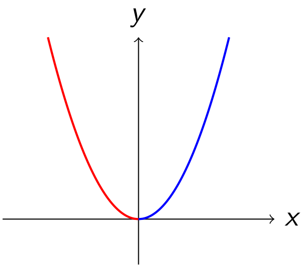
One more term to know: Notice how the graph of $f(x) = x^2$ is symmetric about the y-axis:



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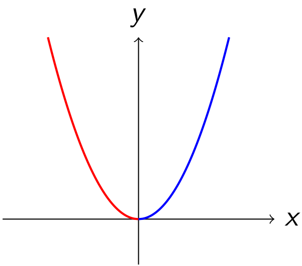

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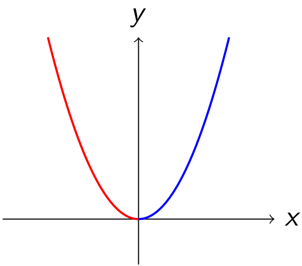

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Definition

We say that f is an **even** function, if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

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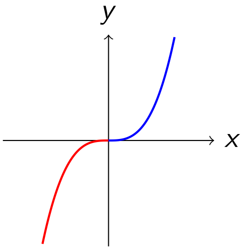

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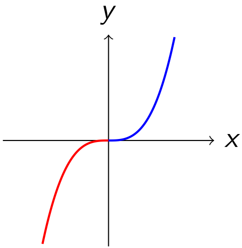

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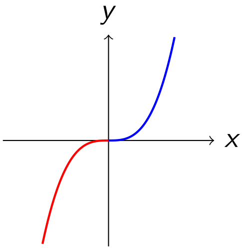

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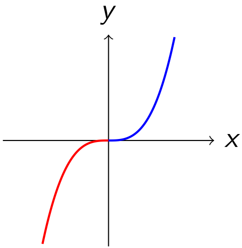
Sometimes, the graph is symmetric about the point $(0, 0)$, e.g.:

$$f(x) = x^3$$



Or, in other words, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Definition

We say that f is an **odd** function, if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Note that a function can be neither even nor odd, e.g. $f(x) = x + 1$.