Conv. mode

LLN

CLT

Відео

$X(w)$ $[0,1]$

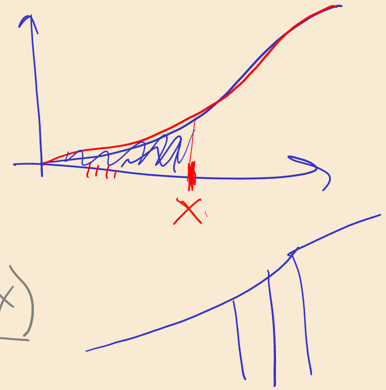
A.s lim

$X_n \xrightarrow{a.s.} X$

$X_n \xrightarrow{d} X$

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

$P(X_n \le x)$



$X \sim N(0,1)$

$Y = -X$ $N(0,1)$

$|X_n(w) - Y(w)|$ $2|X(w)|$

$P(\{w \in \Omega \mid \lim_{n\to\infty} X_n(w) = X(w)\}) = 1$

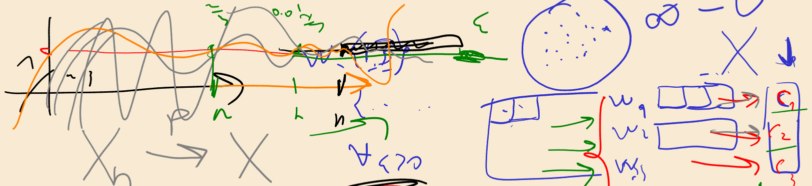
SLLN

w_0 $\forall n$ X

w_1

w_2

$\frac{w}{\infty} = 0$



$X_n \to X$

$\forall \varepsilon > 0$

$X_1, X_2, \ldots$ $E x = \mu$ V, $\sigma^2 < \infty$

$\bar{X}_n \xrightarrow{P} \mu$

$\mathrm{Var}(X_1) = \sigma^2$

$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$\frac{1}{n^2}\mathrm{Var}\left(\sum X_i\right)$

$\frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$

$\frac{\sigma^2}{n \cdot \varepsilon^2}$

$P(|\bar{X}_n - \mu| \ge \varepsilon) \to 0$

X_1, X_2, X_3

$\lim_{n\to\infty} P(\{w \in \Omega \mid |X_n(w) - X(w)| \ge \varepsilon\}) = 0$

$w \in \Omega$ $\lim X_n(w) = X(w)$

$w \to 0$

$n \to$ $n = 4$ 100%

w_0 v v

w_1 v x

w_2 x v

$X_n \to X$

$\mathrm{Var}(aX) = a^2$

$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$

$E[X]$ σ^2

$\boxed{P(|X - \mu| \ge \delta) \le \frac{\sigma^2}{n\delta^2}} \to 0$

$X_1 \to \mu$

$\frac{1}{n}\sum_{i=1}^{n} X_i$

$\frac{1}{n\varepsilon^2}\sigma^2$

SLLN

$E|X_i| <$

$\bar{X}_n \to \mu$

🧀 🧀 01 Convergence modes: The vanishing spike

Define X_n as follows: with probability $\frac{1}{n}$, $X_n = n$; otherwise $X_n = 0$.

- a. Compute $E[X_n]$ and $\mathrm{Var}[X_n]$. What happens as $n \to \infty$?
- b. Show that $X_n \xrightarrow{P} 0$ (converges in probability to 0) by computing $P[|X_n| > \epsilon]$ for any $\epsilon > 0$.
- c. Does $X_n \to 0$ almost surely? Hint: Consider $\sum_{n=1}^{\infty} P[X_n \neq 0]$. Use the Borel-Cantelli lemma intuition: if events happen "infinitely often" then convergence a.s. fails.
- d. Explain in 2-3 sentences: why can a sequence have vanishing probability of being far from 0, yet still "spike" infinitely often with probability 1?

$X_1, X_2, \dots$ iid μ, $0 < \sigma^2 < \infty$

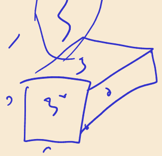
$\bar{X}_n \approx N(\mu, \frac{\sigma^2}{n})$

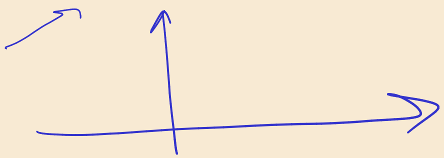
$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$

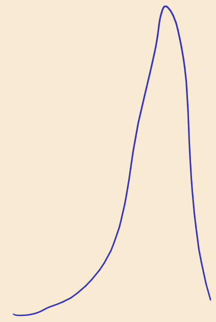
X_n $\quad \frac{1}{n} \to n$, else 0

X_1

$\frac{1}{3}$, 3, 3^2







$E[X_n] = n\frac{1}{n} + 0(1 - \frac{1}{n}) = 1$

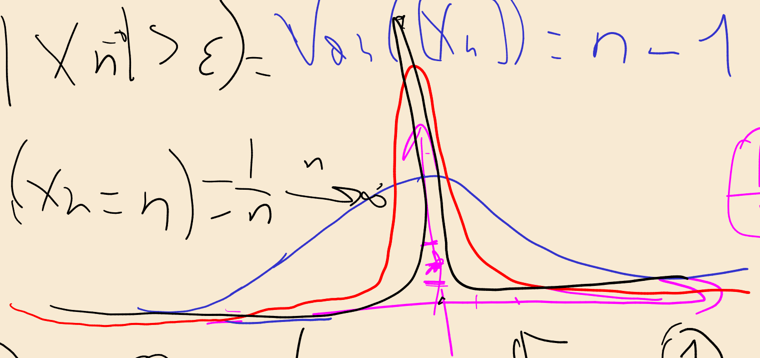
$\frac{1}{n}$ $\quad$ 0

$E[(X_n)^2] = n^2 \cdot \frac{1}{n} + \quad = n$

$P(|X_n| > \varepsilon) = \mathrm{Var}(X_n) = n - 1^2 = n - 1$

$= P(X_n = n) = \frac{1}{n} \xrightarrow{n} 0$

$\frac{1}{n} \to n$



a.s. $\Rightarrow$ P $\Rightarrow$ d

E $\to$ 1 $\quad \frac{1}{n} \to n$

$A_n = \{X_n = n\}$

$P(A_n) = \frac{1}{n}$

$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

$P(A_n$ infinitely often$) = 1$

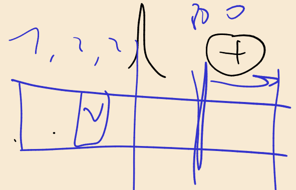
The Infinite Monkey Theorem

Claim: A monkey typing randomly will eventually type the complete works of Shakespeare, almost surely.

Proof sketch:

- Let A_n = “the monkey types ‘To be or not to be’ starting at position n.”
- $P[A_n] = (1/26)^{18} \approx 10^{-25}$ (assuming 26 letters)
- $\sum P[A_n] = \infty$ (sum of constants diverges)
- Keystrokes are independent $\Rightarrow$ BC2 says it happens infinitely often!
- Same argument works for the full text of Hamlet (just a longer string with smaller $P[A_n]$, but still divergent sum).

$A_1, A_2, A_3, A_4 \ldots$



A_n

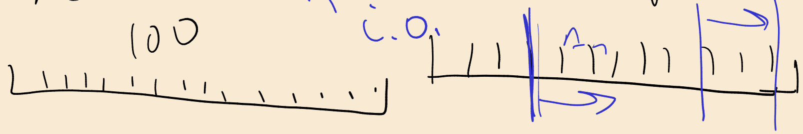
$\sum_{n=1}^{\infty} P(A_n) < \infty$ i.o.

$\bigcap \bigcup A_{ij}$ 1000s

events

limsup

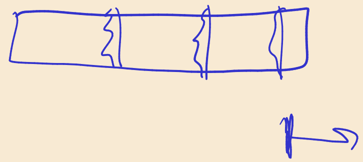
100



$P($ all $\ldots$ A_n $\ldots) = 0$

$\frac{1}{n^2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$

$\sum_{n=1}^{\infty} \frac{1}{k} = \infty$



$A_n \quad \sum_{n=1}^{\infty} P(A_n) = \infty$

$P(\bigcup A_n) = 1$

$P(A_n) = \left(\frac{1}{26}\right)^{1{,}000{,}000}$ ∞ $\frac{1}{n}$

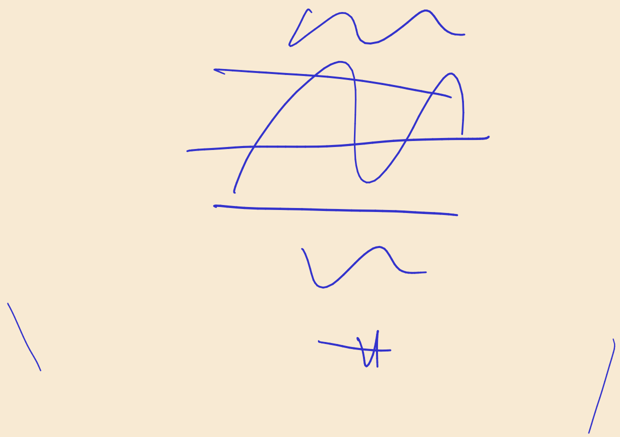
🧀 02 Sample mean convergence: Visualizing LLN

You roll a fair die repeatedly and compute the running average $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

- a. What is $E[X_i]$ and $\mathrm{Var}[X_i]$ for a single roll?
- b. Compute $E[\bar{X}_n]$ and $\mathrm{Var}[\bar{X}_n]$. What happens to the variance as n increases?
- c. Use Chebyshev's inequality to bound $P[|\bar{X}_n - 3.5| > 0.5]$. For what n is this probability less than 0.05?
- d. Sketch (or describe) how you'd expect a plot of $\bar{X}_n$ vs n to look for $n = 1$ to $n = 100$.

$P(\bar{X}_n - 3.5 > 0.5)$

0.05

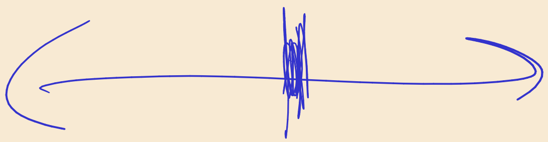


$n \geq 233$

CLT

$[3, 4]$

$\pm\%$



$P(X_n - 3.5) \rightarrow$

$\frac{1}{1000}$

$n = 234$

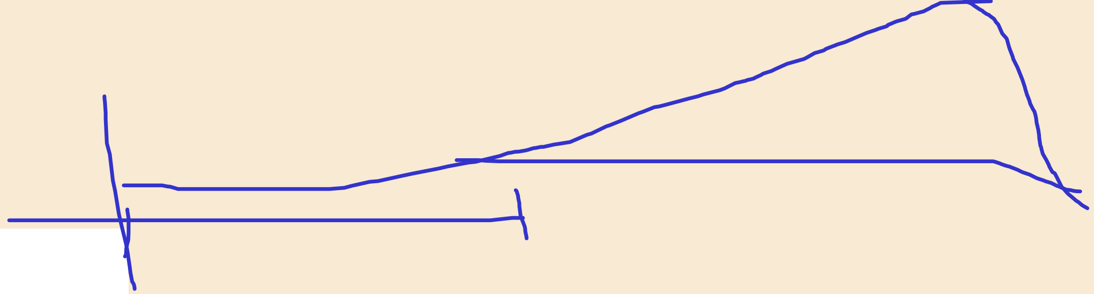
$0.0.0005$

🧀 🧀 🧀 03 When LLN fails: Cauchy counterexample

The Cauchy distribution has PDF $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$.

- a. Without computing, explain why $E[X]$ doesn't exist for $X \sim \text{Cauchy}$ by considering the integral $\int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1+x^2)} dx$.
- b. Let $X_1, \ldots, X_n$ be i.i.d. Cauchy. The sample mean is $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Does $\bar{X}_n$ converge to anything as $n \to \infty$?
- c. **Fact:** $\bar{X}_n$ itself has the *same* Cauchy distribution for all n. Explain why this violates the LLN, and what condition of the LLN is not satisfied.
- d. Simulate (or imagine) 1000 samples from Cauchy and compute their mean. Would you expect a "tight" estimate or wild variability? Why is averaging useless here?

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2}$$



🧀 🧀 04 CLT in Python: Simulating the magic

In this problem, you'll write Python code to **demonstrate the Central Limit Theorem** empirically.

Setup: Let $X_i \sim \text{Exp}(1)$ (exponential distribution with rate 1, so $E[X_i] = 1, \text{Var}[X_i] = 1$).

Tasks:

- a. **Generate data:** For each sample size $n \in \{1, 5, 10, 30, 50\}$:
  - Simulate $N = 10000$ sample means $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ (each mean computed from n i.i.d. $\text{Exp}(1)$ random variables).
  - Store these 10000 sample means.
- b. **Standardize:** Compute the standardized version:

$$Z_n = \frac{\bar{X}_n - 1}{\sqrt{1/n}} = \sqrt{n} \cdot (\bar{X}_n - 1).$$

  For each n, compute this for all 10000 samples.
- c. **Visualize:** Create a figure with 5 subplots (one for each n):
  - Plot a **histogram** of the 10000 standardized values Z_n.
  - Overlay the **standard normal PDF** $N(0, 1)$ as a smooth curve.
  - Add a title indicating the sample size n.
- d. **Interpret:**
  - For which values of n does the histogram closely match the normal curve?
  - The original $\text{Exp}(1)$ distribution is **highly skewed** (right tail). Explain in 2-3 sentences why the CLT "fixes" this skewness as n grows.

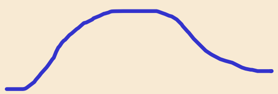
$\text{Exp}(1)$

$\mu = X = 1$

$\sigma^2 = 1$

CLT

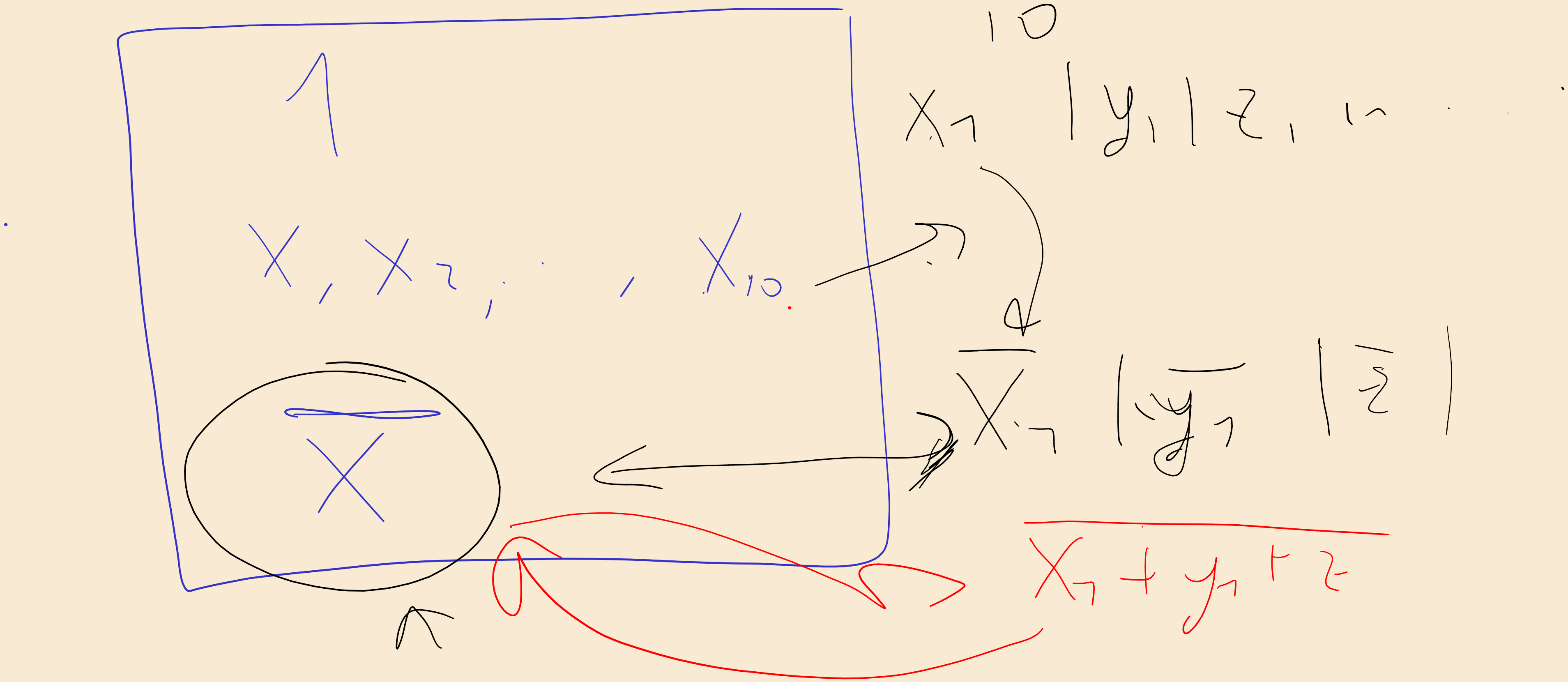
$\bar{X}_n \sim N$



X_1 X_2 . . . . X_{1n}

10 10 30

$\bar{x}_1$ $\bar{x}_2$


1
$X_1, X_2, \ldots, X_{10}$
$\bar{X}$
10
$X_1 \mid y_1 \mid z_1 \ldots$
$\bar{X}_1 \mid \bar{y}_1 \mid \bar{z} \mid$
$\overline{X_1 + y_1 + z}$