1) Expectation & Variance Basics

🧀 01 Modified Die: Probability and Moments

Vahe added a dot on the 4 side of the die, making it 5, and then added two dots on the 1 side, making it 3.

- a. What is the probability that the outcome of the die is greater than 4?
- b. Find the expectation and variance of the modified die.

🧀 02 Die Game: Expected Value

You roll a fair die. If you roll 1, you are paid $25. If you roll 2, you are paid $5. If you roll 3, you win nothing. If you roll 4 or 5, you must pay $10, and if you roll 6, you must pay $15.

- a. Compute the expected payoff.
- b. Do you want to play?

🧀 🧀 🧀 03 Uniform Sum Expectation

Let X and Y be two continuous random variables with uniform distribution on $(0, 2)$.

- Find $\mathbb{E}[X + Y]$.

$\mathbb{E}$ 4 $\frac{3}{6} = \frac{1}{2}$

1 2 3 4 5 6

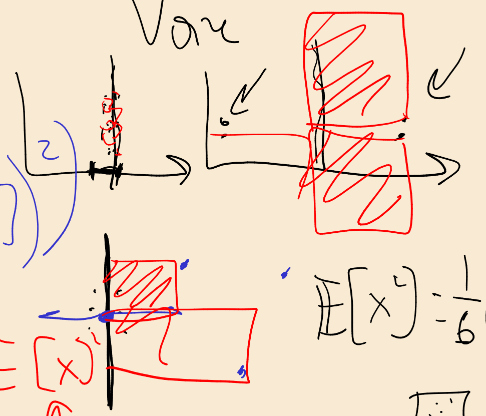
3 2 3 5 5 6

x_1 x_2 x_3 x_4 x_5

$\sum_{i=1}^{6} p(x_i) \cdot x_i$

$\frac{1}{6}(3+2+3+5+5+6)$

Var



$\mathbb{E}((X - \mathbb{E}(x)))$

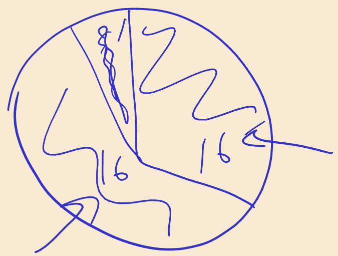
$\mathbb{E}[x^2] = \frac{1}{6}(3^2+2^2+3^2+5^2+5^2+6^2)$

1 2 3 4 5 6

25 5 0 −10 −10 −15

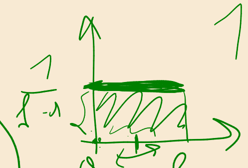
$$\mathbb{E}[X] = \frac{25+5+0-10-10-15}{6} = -\frac{5}{6} \approx -0.833$$

$\mathrm{Var} = \mathbb{E}[X^2] - \mathbb{E}[x]^2$



16 17

$\frac{16-17}{33} =$



$f(x) = \frac{1}{b-a}$

$X, Y \sim U(0, 2)$

$\mathbb{E}(X+Y) = \mathbb{E}[X] + \mathbb{E}[Y]$ 0-1 { 1-2

$Var(X+Y) = Var(X) + Var(X)$ $E(X) = E(X)\frac{0+2}{2}$

cov

🧀 🧀 04 Expectation Without the CDF

Let $X \sim \mathrm{Uniform}(0,1)$. Define $Y = \log(1+X)$.

- a. Compute $\mathbb{E}[Y]$ using LOTUS directly.
- b. Compute $\mathrm{Var}(Y)$ (you may leave integrals in closed form).

$\mathbb{E}(X) = \int x f(x) dx$

$f(x) = \frac{1}{1-0} = 1$ $E(Y) = \int g(x) f(x) dx = \int_0^1 \log(1+x) dx$

$u = 1+x$

$\int_1^2 \log u \, du$

🧀 🧀 05 Piecewise Payoff

Let $X \sim \mathrm{Exp}(\lambda)$. A "refund policy" pays $g(X) = \min(X, c)$ for fixed $c > 0$.

(for $\mathrm{Exp}(\lambda)$, $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Aprikyan will cover distibutions during next or next-next lesson.)

- a. Compute $\mathbb{E}[g(X)]$ using LOTUS.
- b. Compute $\mathbb{P}(g(X) = c)$.
- c. Find $\frac{d}{dc}\mathbb{E}[g(X)]$ and interpret.

$X \sim Exp(\lambda)$ $g(x) = \min(X, c)$ $c > 0$

$\mathbb{E}(g(x)) = \int_0^\infty \min(x, c) \lambda e^{-\lambda x} dx =$

$\mathbb{P}(X<c)$

$= \int_0^c x \lambda e^{-\lambda x} dx + \int_c^\infty c \cdots$ $\mathbb{P}(g(x) = c) = \mathbb{P}(X \geq c) =$

$= 1 - P(X<c) = 1 - (1 - e^{-\lambda c})$

06 When to Stop (Secretary-lite) √

You see prices of used laptops one by one, i.i.d. $\mathrm{Uniform}(0, 1)$. You can accept one price and stop, or reject and continue; once rejected, it's gone. You must decide a stopping rule.

- a. Consider the rule: "accept the first price $\leq t$." Compute the expected accepted price as a function of t given a maximum of N offers.
- b. Find (approximately) the best t for $N = 10$.

07 Optimal Reroll (Single Reroll Allowed)

You roll a die once; you may choose to keep it or reroll once (then must keep). Goal: maximize expected value.

- a. What threshold rule is optimal?
- b. What is the resulting expected value?
- c. Compute the variance of the final payoff under the optimal strategy.

$\boxed{3.5}$

4 5 6 → 5

$$\underbrace{P(X \geq 4)\,E[X \mid X \geq 4]}_{\frac{1}{2}\cdot 5} + \underbrace{P(X \leq 3)\,E(Y)}_{\frac{1}{2}\cdot 3.5} = \frac{17}{4}$$

3.5~

08 St. Petersburg Game (Bonus)

A fair coin is tossed until the first Heads appears. If Heads appears on toss k, you get 2^k dollars.

- a. Compute the expected payoff.
- b. Why might people still refuse to pay an "infinite fair price" to play?

H K 2^K

1000 900

$$\left(\frac{1}{2}\right)^{K-1}\cdot\frac{1}{2} = \left(\frac{1}{2}\right)^K \qquad \sum_{k=1}^{\infty}\frac{1}{k} = \sim 12$$

$$\sum_{k=1}^{\infty} 2^k \left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} 1 = +\infty$$

🧀 🧀 11 Markov — “What’s the chance my bill is huge?”

Your monthly electricity bill $B \geq 0$ has average $\mathbb{E}[B] = \$80$.

- a. Use Markov’s inequality to bound $\mathbb{P}(B \geq \$200)$ and $\mathbb{P}(B \geq \$300)$.
- b. Suppose the provider claims: “the probability of a $\$300+$ bill is at most 5%.” What average bill $\mathbb{E}[B]$ would make this statement true by Markov?

🧀 🧀 12 Chebyshev — “Commute-time reliability”

Commute time T (minutes) has mean $\mu = 40$ and variance $\sigma^2 = 25$ (so $\sigma = 5$).

- a. Use Chebyshev’s inequality to upper-bound $\mathbb{P}(T \geq 55)$ and $\mathbb{P}(T \leq 25)$.
- b. How large must a time buffer b be so that $\mathbb{P}(T \leq \mu + b) \geq 0.95$?

$P(X \geq 300) \leq 0.05$

$\frac{E[X]}{300} \leq 0.05$

$E[X] \leq 15$

$B \geq 0$ $\quad \mathbb{E}[B] = 80$

$X \geq 0$ $\quad a > 0$

$P(X \geq a) \leq \frac{E[X]}{a}$

$P(X \geq 200) \leq \frac{80}{200} = 0.4$

$P(|X - \mathbb{E}(X)| \geq k) \leq \frac{\sigma^2}{k^2}$

$(X - \mu)^2$

$\mathbb{E}[X] = 40$

T 40, 25

$P(T \geq \mathbb{E}[T] + k) \leq 0.05$

$P(|T - 40| \geq k) \leq \frac{\sigma^2}{k^2} \leq 0.05$

$\frac{25}{k^2} \leq 0.05 \quad k \geq 22$

$P(T \geq 55) = P(|T - 40| \geq 15) = \frac{25}{15^2} = \frac{25}{225} = \frac{1}{9}$

Cantel:

$P(T - \mathbb{E}[X] \geq k) \leq \frac{\sigma^2}{\sigma^2 + k^2}$

Consider a family that has two children. The sample space of genders is S = {(G,G), (G,B), (B,G), (B,B)} where G denotes a girl and B a boy, and all outcomes are equally likely.

- a. What is the probability that both children are girls, given that the first child is a girl?
- b. Suppose the father answers “Yes” to “Do you have at least one daughter?”. Given this information, what is the probability that both children are girls?

GG, GB, BG, ~~BB~~

$P(GG \mid G) = \frac{1}{2}$

$P(GG \mid \text{at least one } G) = \frac{1}{3}$

🧀 🧀 20 Two Children, One Named Lilia

A family has two children. We ask the father: “Do you have at least one daughter named Lilia?”, and he replies “Yes.” What is the probability that both children are girls?

Assume:

- If a child is a girl, her name is Lilia with probability α < 1, independently of other children’s names.
- If the child is a boy, his name will not be Lilia.

$\alpha \approx 0$

$\alpha = 1$

$\frac{2}{4} \to \frac{2-1}{4-1} = \frac{1}{3}$

$\alpha = 0$

α

$$= \frac{2-\alpha}{4-\alpha} = \frac{GG}{GG + GB + BG + BB} = \frac{\frac{1}{4}(2\alpha - \alpha^2)}{\frac{1}{4}(2\alpha - \alpha^2) + \frac{1}{4}\alpha + \frac{1}{4}\alpha + 0} = \frac{2-\alpha}{2-\alpha+2}$$

$P(\text{Lilia} \mid BB) = 0$

$P(\text{Lilia} \mid GG) = (1-(1-\alpha)^2)\frac{1}{2}\cdot\frac{1}{2}$

$P(\text{Lilia} \mid GB) = \frac{1}{2}\cdot\frac{1}{2}\cdot\alpha = P(\text{Lilia} \mid BG)$

$\frac{1}{4}(1 - 1 + 2\alpha - \alpha^2)$

$= \frac{1}{4}(2\alpha - \alpha^2)$